JOHN W. WOLL, JR.

Functions of Several Variables

HARBRACE COLLEGE MATHEMATICS SERIES

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HARBRACE COLLEGE MATHEMATICS SERIES

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Foreword

The Harbrace College Mathematics Series has been undertaken in response to the growing demands for flexibility in college mathematics curricula. This series of concise, single-topic textbooks is designed to serve two primary purposes: First, to provide basic undergraduate text materials in compact, coordinated units. Second, to make available a variety of supplementary textbooks covering single topics.

To carry out these aims, the series editors and the publisher have selected as the foundation of the series a sequence of six textbooks covering functions, calculus, linear algebra, multivariate calculus, theory of funtions, and theory of functions of several variables. Complementing this sequence are a number of other planned volumes on such topics as probability, statistics, differential equations, topology, differential geometry, and complex functions.

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SALOMON BOCHNER

W. G. LISTER

Preface

This book is an exposition of selected topics from the calculus of functions of several variables. It is intended for undergraduate mathematics students in the third or fourth year analysis program, who have had several semesters of the calculus and at least an introduction to linear algebra.

Specifically, the prerequisites include knowledge of the real numbers and functions of one variable plus some introductory experience with multivariate calculus of the type that is usually included in the first two years of college mathematics. The linear algebra needed, which is approximately the content of *Linear Algebra* by Ross A. Beaumont, includes the concept of a finite dimensional vector space, some experience with the idea of a basis for a vector space, and some elementary concepts and properties associated with linear transformations, such as those of rank and determinants. Aside from the fact that the fundamental existence and uniqueness theorem for ordinary differential equations is used without proof, the results used are proved in the body of the text.

The topics treated in this book were selected with two primary objectives: (1) these topics cover the notions usually referred to as "vector analysis," and (2) they cover concepts that can be easily generalized to differentiable manifolds in a relatively coordinate-free manner.

The book divides naturally into three sections. The first two chapters are rather standard, treating respectively the point set topology of $\mathbf{R}^n$ and differentiation on $\mathbf{R}^n$. In the second chapter the inverse function theorem and the theorem on change of variables in multiple integrals are proved and several important implications are discussed in detail. The latter include the concept of local coordinates and the rank of a differentiable map from $\mathbf{R}^m$ to $\mathbf{R}^n$. Basically preparatory, these two chapters constitute the theoretical foundations of the material developed in the remainder of the book.

Chapters Three, Four, and Five constitute the next unit. They are basically manipulative. In Chapter Three the notion of a (tangent) vector at $p \in \mathbf{R}^n$ and the dual notion of covectors at p are developed. With this introduction, Chapter Four is devoted to exposition of the multilinear algebra necessary to construct and verify the properties of exterior multiplication. This chapter actually includes a little more than is needed, however, since the exterior product is constructed by antisymmetrization of multilinear forms rather than by the somewhat more elementary method of giving a multiplication table with respect to a specific basis and showing that the resulting properties of the product imply uniqueness. Chapter Five treats differential forms on $\mathbf{R}^n$, k-chains, Stokes theorem, and some related integral expressions involving the metric, such as Green's identities and Poisson's integral formula for harmonic functions.

Chapter Six treats the concept of a flow with velocity field X and the related derivations on vector fields and differential forms. It includes Frobenius' theorem on completely integrable systems of first-order partial differential equations and Poincaré's lemma that a closed differential form is locally exact.

Chapter Seven shows how the notation and ideas developed earlier can be used in the theory of functions of a complex variable. After a discussion of terminology and of the concept of an analytic coordinate system in the first two sections, the remainder of the chapter is devoted to developing some of the standard material centering around Cauchy's integral formula and power series expansions. The nature of these last two chapters is again somewhat more theoretical than manipulative.

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Contents

CHAPTER ONE

Topology of $\mathbf{R}^n$

1 Fundamental structure of $\mathbf{R}^n$

n-dimensional euclidean space $\mathbf{R}^n$ is the set of all n-tuples $p = (a^1, \ldots, a^n)$ of real numbers, a^k represents the kth member of the n-tuple (not a to the kth power), and the letters p and q are used to represent elements of $\mathbf{R}^n$. $\mathbf{R}^n$ is a vector space, two points $p = (a^1, \ldots, a^n)$ and $q = (b^1, \ldots, b^n)$ having the sum $p + q = (a^1 + b^1, \ldots, a^n + b^n)$. If λ is a real number, $\lambda p = (\lambda a^1, \ldots, \lambda a^n)$. The length of an element $p = (a^1, \ldots, a^n)$ of $\mathbf{R}^n$ is given by

$$\|p\| = \{\sum_{k=1}^{n} (a_k)^2\}^{1/2}$$

and it satisfies the important relations

$$\|p + q\| \leq \|p\| + \|q\|;$$
$$\|\lambda p\| = |\lambda| \, \|p\|.$$

The euclidean distance $d(p, q)$ between p and q is the length $\|p - q\|$.

2 Open sets, closed sets, and neighborhoods

The ε-*ball centered at* q or, equivalently, the *ball of radius* ε *centered at* q is the subset $B_\varepsilon(q)$ of $\mathbf{R}^n$ consisting of those points p for which $d(p, q) \leq \varepsilon$.

$$B_\varepsilon(q) = \{p \in \mathbf{R}^n : d(p, q) \leq \varepsilon\}.$$

q is an *interior point* of the set A if A contains some ball of positive radius centered at q as a subset. The set of interior points of A is denoted by interior (A). The set U is *open* if U = interior (U), while U is a *neighborhood of* q if $q \in$ interior (U). So a set is open if and only if it is a neighborhood of each of the points it contains. The empty set $\varnothing$ does not contain any points and accordingly is equal to its own interior and open. Every point of $\mathbf{R}^n$ is an interior point of $\mathbf{R}^n$, so that $\mathbf{R}^n$ is also open.

A set F is *closed* if its complement F^c, the set of points in $\mathbf{R}^n$ which are not members of F, is open. For example, the complement of the empty set $\varnothing$ is $\varnothing^c = \mathbf{R}^n$ which is open, so that $\varnothing$ is closed. ($\varnothing$ is both open and closed as is $\mathbf{R}^n$.)

A few of the more important properties of open and closed sets are established below as examples; many other properties are left as exercises. In general, of course, most sets are neither open nor closed.

(2.1) Example. If $U_1, \ldots, U_m$ are open sets, their intersection $U_1 \cap U_2 \cap \cdots \cap U_m$ is open. In fact, if p belongs to $U_1 \cap \cdots \cap U_m$, then for each $i = 1, \ldots, m$ there is a number $\varepsilon_i > 0$ such that $p \in B_{\varepsilon_i}(p) \subset U_i$. The intersection of these concentric balls $B_{\varepsilon_i}(p)$ is the ball $B_\delta(p)$ where δ = minimum $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m\}$, so that $p \in B_\delta(p) \subset B_{\varepsilon_i}(p) \subset U_i$ for each i. That is, $B_\delta(p) \subset U_1 \cap \cdots \cap U_m$ and p belongs to interior $(U_1 \cap \cdots \cap U_m)$. Since p was an arbitrary point of $U_1 \cap \cdots \cap U_m$, this intersection is open.

(2.2) Example. The union of an arbitrary class of open sets is open. Let $\{U_\alpha : \alpha \in \Gamma\}$ be a class of open sets and let S be the union of all the sets U_α, $\alpha \in \Gamma$. If $p \in S$, then $p \in U_\beta$ for some $\beta \in \Gamma$, and since U_β is open, $B_\varepsilon(p) \subset U_\beta \subset S$ for some $\varepsilon > 0$. Thus $p \in$ interior (S) and S is open.

(2.3) Example. The intersection of any class of closed sets is closed. Let $\{F_\alpha : \alpha \in \Gamma\}$ be a class of closed sets whose intersection is D. A point belongs to each F_α if and only if it does not belong to any of the sets F_α^c, $\alpha \in \Gamma$. $S = \bigcup_\alpha F_\alpha^c$ is known to be open by Example (2.2) and D is consequently closed.

A subset V of the set D in $\mathbf{R}^n$ is called *relatively open in* D if for each $p \in V$ there is a ball $B_\varepsilon(p)$ centered at p such that $B_\varepsilon(p) \cap V = B_\varepsilon(p) \cap D$. Corresponding to this, a subset F of the set D in $\mathbf{R}^n$ is called *relatively closed in* D if $D \cap F^c$ is relatively open in D. The consequences of these definitions are left to the exercises.

Exercises

2.1 Show that V is relatively open in D if and only if $V = D \cap W$ for some open subset W in $\mathbf{R}^n$.

2.2 Give an example of a set which is neither closed nor open.

A sequence of sets $A_1, A_2, \ldots, A_n, \ldots$ is *monotone decreasing* or just decreasing if $A_1 \supset A_2 \supset \cdots \supset A_n \supset A_{n+1} \supset \cdots$.

2.3 Give an example of a sequence of open sets $\{U_n\}$ which is decreasing and has empty intersection, $\bigcap_{n=1}^{\infty} U_n = \varnothing$.

2.4 Give an example of a sequence of closed sets $\{F_n\}$ which is decreasing and has empty intersection, $\bigcap_{n=1}^{\infty} F_n = \varnothing$.

2.5 Show that the intersection of a countable number of open sets can be closed; can be a set which is neither open nor closed; can be open.

2.6 Show that the union of a countable number of closed sets can be a set which is neither open nor closed.

2.7 Show that if $F_1, \ldots, F_k$ are closed subsets of $\mathbf{R}^n$, then so is $F_1 \cup \cdots \cup F_k$. A finite union of closed sets is closed.

2.8 For each subset $F \subset \mathbf{R}^n$ let cl (F) denote the intersection of all the closed subsets of $\mathbf{R}^n$ which contain F. cl (F) is closed by Example (2.3).

(a) Show that cl (F) is the smallest closed subset of $\mathbf{R}^n$ which contains F.
(b) Show cl $(F) = F$ if and only if F is already closed. In particular cl (cl (F)) = cl (F) for any subset F.
(c) Show cl $(F \cup E) \supset$ cl $(F) \cup$ cl (E) for any E and F in $\mathbf{R}^n$.

2.9 Let $B^\circ_\varepsilon(q) =$ interior $B_\varepsilon(q) = \{p \in \mathbf{R}^n : d(p, q) < \varepsilon\}$ be the open ε-ball centered at q. $q = (a^1, \ldots, a^n)$ is called a *rational point* if each of its coordinates a^k is a rational number.

(a) Show that the set of rational points of $\mathbf{R}^n$ is countable.
(b) Let $\mathcal{U} = \{B^\circ_\varepsilon(q) : \varepsilon$ is a rational number and q is a rational point of $\mathbf{R}^n\}$. Show that the class of subsets $\mathcal{U}$ of $\mathbf{R}^n$ is countable.
(c) If p is an interior point of E, show that there is a set $B^\circ_\varepsilon(q) \in \mathcal{U}$ such that $p \in B^\circ_\varepsilon(q) \subset E$.
(d) Show that each open set J in $\mathbf{R}^n$ can be expressed as a (countable) union of sets in the class $\mathcal{U}$.

2.10 Show that the class of open subsets of $\mathbf{R}^n$ is not countable.

2.11 (Based on Exercise 2.9.) If $\mathcal{V}$ is a class of open subsets of $\mathbf{R}^n$, let $V = \cup \{W : W \in \mathcal{V}\}$ be the union of all the sets in $\mathcal{V}$. Show that $\mathcal{V}$ has a countable subclass $\mathcal{V}_0 \subset \mathcal{V}$ such that $V = \cup \{W : W \in \mathcal{V}_0\}$.

A set E is called a G_δ ("G-delta") if it can be expressed as the intersection of a countable number of open sets. $E = \bigcap_{k=1}^{\infty} V_k$, where V_k is open.

2.12 Show that the unit ball $B_1((0, \ldots, 0))$ centered at the origin is a G_δ.

For each closed set $F \subset \mathbf{R}^n$ and each $p \in \mathbf{R}^n$ put

$$d(p, F) = \inf \{d(p, q) : q \in F\}.$$

2.13 Show that whenever F is closed and $\varepsilon > 0$ the set $F_\varepsilon = \{q : d(q, F) < \varepsilon\}$ is open.

2.14 (Based on Exercise 2.13.) Show that $F = \bigcap_{k=1}^{\infty} F_{1/k}$. That is, show that each closed subset of $\mathbf{R}^n$ is a G_δ.

2.15 For each subset $A \subset \mathbf{R}^n$ let $U(A) = \{q \in A : B_\varepsilon(q) \cap A$ is at most countable for some $\varepsilon > 0\}$. Show $U(A)$ is a countable subset of A.

A class $\mathfrak{U}$ of subsets of $\mathbf{R}^n$ is a *covering of B* or covers B if $\cup \{U : U \in \mathfrak{U}\} \supset B$. $\mathfrak{V}$ is a *subcovering* of the preceding covering if $\mathfrak{V} \subset \mathfrak{U}$ and $\mathfrak{V}$ covers B. $\mathfrak{V}$ is a *finite subcovering* or *countable subcovering* of B if the class $\mathfrak{V}$ is finite or countable, respectively.

2.16 Let $\mathfrak{U}$ be a covering of B and suppose each $U \in \mathfrak{U}$ is open. Show that $\mathfrak{U}$ has a countable subcovering $\mathfrak{V}$ of B. [HINT: See Exercises 2.9 and 2.11.]

3 Sequences

A sequence $p_1, p_2, \ldots, p_k, \ldots$ of points of $\mathbf{R}^n$ *converges* to the point p—in symbols, $\lim_k p_k = p$ or $\lim p_k = p$—if and only if for each neighborhood V of p the set $\{k : p_k \notin V\}$ is finite. In this case p is called the *limit* of the sequence $\{p_m\}_{m=1}^{\infty}$. (The reader is cautioned that the points p_j need not differ for different values of j. It is even perfectly possible that $p_1 = p_k$ for all k.) Since each $B_\varepsilon(p)$ is a neighborhood of p, $\lim p_m = p$ if and only if for each $\varepsilon > 0$, $\|p_k - p\| < \varepsilon$, except possibly for a finite number of k's. Stated alternatively, $\lim p_m = p$ if and only if the limit of the numerical sequence $\{\|p_k - p\|\}$ is zero. The concept of convergence can be phrased in another manner. The sequence $\{p_k\}_{k=1}^{\infty}$ is *ultimately in the set J* if and only if $\{k : p_k \in J\}$ is finite. In this terminology a sequence $\{p_m\}_{m=1}^{\infty}$ converges to the point p if and only if it is ultimately in each neighborhood of p.

$\mathbf{R}^n$ is the space of n-tuples of real numbers, so that a sequence whose kth term is $p_k = (a_k^1, \ldots, a_k^n)$ gives rise to n sequences of real numbers $\{a_k^1, k = 1, 2, \ldots\}, \ldots, \{a_k^n, k = 1, 2, 3, \ldots\}$. The inequalities

$$(3.1) \qquad \max \{|a_k^1 - a^1|, \ldots, |a_k^n - a^n|\} \le \|p_k - p\| \le n \max \{|a_k^1 - a^1|, \ldots, |a_k^n - a^n|\}$$

show that $\lim p_k = p$ where $p = (a^1, \ldots, a^n)$ if and only if $\lim_k a_k^j = a^j$ for each $j = 1, 2, \ldots, n$. This last observation can be exploited to reduce many properties of sequences in $\mathbf{R}^n$ to corresponding properties for sequences of real numbers.

(3.2) Example. $\{p_k\}$ is a *Cauchy sequence* in $\mathbf{R}^n$ if for each $\varepsilon > 0$ the set $\{j : \text{for some } m \ge j, \|p_m - p_j\| > \varepsilon\}$ is finite. Every Cauchy sequence in $\mathbf{R}^n$ converges. In fact an inequality like the first inequality in (3.1) shows that each of the subsidiary sequences $\{a_k^1\}, \ldots, \{a_k^1\}$ is a Cauchy sequence of real numbers. Each Cauchy sequence of real numbers converges (this is one of the basic properties of **R**); so $\lim a_k^1 = a^1, \ldots, \lim a_k^n = a^n$. According to the observation preceding this example, $\lim p_k = p$ where $p = (a^1, \ldots, a^n)$.

The sequence $\{q_m\}$ is a *subsequence* of the sequence $\{p_k\}$ if $q_m = p_{k(m)} (m = 1, 2, \ldots)$ where $m \rightarrow k(m)$ is a map which assigns to each positive integer m another positive integer $k(m)$ subject only to the requirement that $k(m + 1) > k(m)$. The point q is an *accumulation point* of the sequence $\{p_k\}$ if for each neighborhood W of q the set $\{k : p_k \in W\}$ is infinite. The sequence $\{p_k\}$ is *eventually in the set* J if and only if $\{k : p_k \subset J\}$ is infinite. In this terminology q is an accumulation point for $\{p_k\}$ when and only when $\{p_k\}$ is eventually in each neighborhood W of q.

As a further criterion: q is an accumulation point of $\{p_k\}$ if and only if $\{p_k\}$ has a subsequence $\{q_m\}$ which converges to q. To see this suppose first that $\{q_m\}$ is a subsequence of $\{p_k\}$, $q_m = p_{k(m)}$, which converges to q. Then for each neighborhood W of q the set $\{m : q_m \notin W\}$ is finite, consequently $\{k(m) : p_{k(m)} = q_m \in W\}$ is infinite. Since this latter set of $k(m)$'s is a subset of $\{k : p_k \in W\}$, it follows that this set too is infinite and by definition q is an accumulation point of $\{p_k\}$. Conversely if q is an accumulation point of $\{p_k\}$, define inductively

$$k(1) = 1, \qquad k(m) = \inf \left\{ j : j > k(m - 1), \|p_j - p\| < \frac{1}{m} \right\}.$$

With this definition $\|p_{k(m)} - q\| < \dfrac{1}{m}$ for each m; so the subsequence $\{q_m\}$, $q_m = p_{k(m)}$, converges to q.

Exercises

3.1 If $\{q_m\}$ is a subsequence of $\{p_k\}$ and q is an accumulation point of $\{q_m\}$, show that q is an accumulation point of $\{p_k\}$.

3.2 If the set D is not closed, show there is a sequence $\{p_k\}$, $p_k \in D$, which converges to a point $p \notin D$.

3.3 Construct a sequence with no accumulation points.

3.4 Construct a sequence in **R** whose set of accumulation points is the unit interval $[0, 1]$.

3.5 If $\{p_k\}$, $\{q_k\}$ are sequences in $\mathbf{R}^n$ with $\lim_k p_k = p$, $\|q_k - p_k\| < 1/k$, show that $\lim_k q_k = p$.

3.6 Let cl (F) be the set of points q which are limits of sequences $\{q_k\}$, $q_k \in F$, chosen from F.

(a) Show that cl (F) is closed.
(b) Show that cl (F) is the smallest closed set containing F.

3.7 Show that the set of accumulation points of a sequence is closed.

3.8 If F is a closed set in $\mathbf{R}$, construct a sequence whose set of accumulation points is F.

A subset $P \subset \mathbf{R}^n$ is *perfect* if (i) P is closed in $\mathbf{R}^n$, (ii) each $p \in P$ is the limit of a sequence $\{q_n\}$ of points $q_n \in P$, $q_n \neq p$, chosen from P with the point p itself removed, (iii) $P \neq \varnothing$, P is not empty.

3.9 Construct examples of perfect and nonperfect closed sets.

3.10 A perfect subset of $\mathbf{R}^n$ is never countable. Suppose P is perfect and countable, $P = \{p_1, p_2, \ldots, p_k, \ldots\}$.

(a) Show for some $\varepsilon_1 > 0$ there is a closed ball B_1 of radius ε_1 which contains infinitely many points of P but does not contain p_1.

(b) Show that if B_k is a closed ball of radius $\varepsilon_k > 0$ which contains infinitely many points of P but does not contain the points $p_1, p_2, \ldots, p_k$, there is a ball $B_{k+1} \subset B_k$ of radius ε_{k+1}, $0 < \varepsilon_{k+1} < \varepsilon_k$, which contains infinitely many points of P but does not contain $p_1, p_2, \ldots, p_{k+1}$.

(c) Suppose the closed balls $B_1 \supset B_2 \supset \cdots \supset B_k \supset B_{k+1} \supset \cdots$ of radii $\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_k > \varepsilon_{k+1} > \cdots$ have been constructed inductively satisfying the conditions in (a) and (b) above and that $\lim_k \varepsilon_k = 0$. Show that each sequence $\{q_m\}$, $q_m \in B_m \cap P$, is a Cauchy sequence.

(d) Obtain a contradiction by considering $\lim_m q_m$, $\{q_m\}$ as in (c), and thus show P could not have been countable.

3.11 If F is closed in $\mathbf{R}^n$, show that $F = C \cup P$ where C is countable and P is perfect. [HINT: Put $C = U(F)$ where $U(F)$ is defined as in Exercise 2.15.]

4 Compact sets

If $\{p_k\}$ converges to q then q is the only accumulation point of $\{p_k\}$, but in general a sequence which has just one accumulation point need not converge to that accumulation point. There is, however, one important situation where this last statement is true. A subset D of $\mathbf{R}^n$ is *compact* if every sequence of points $\{p_k\}$ of D has an accumulation point in D. Equivalently a subset D of $\mathbf{R}^n$ is compact if every sequence of points $\{p_k\}$ of D has a subsequence which converges to a point of D.

(4.1) Example. If D is compact and p is the only accumulation point of the sequence $\{p_k\}$ of points of D, then $\lim p_k = p$. Suppose on the contrary that for some neighborhood W of p $\{k : p_k \notin W\}$ is infinite. In this case $\{p_k\}$ has a subsequence $\{q_m\}$ with $q_m \notin W$ for each m. $\{q_m\}$ is still a sequence of points in D and hence $\{q_m\}$ has an accumulation point q which (Exercise 3.1) is an accumulation point of $\{p_k\}$. As p is the only accumulation point of $\{p_k\}$, $p = q$. This