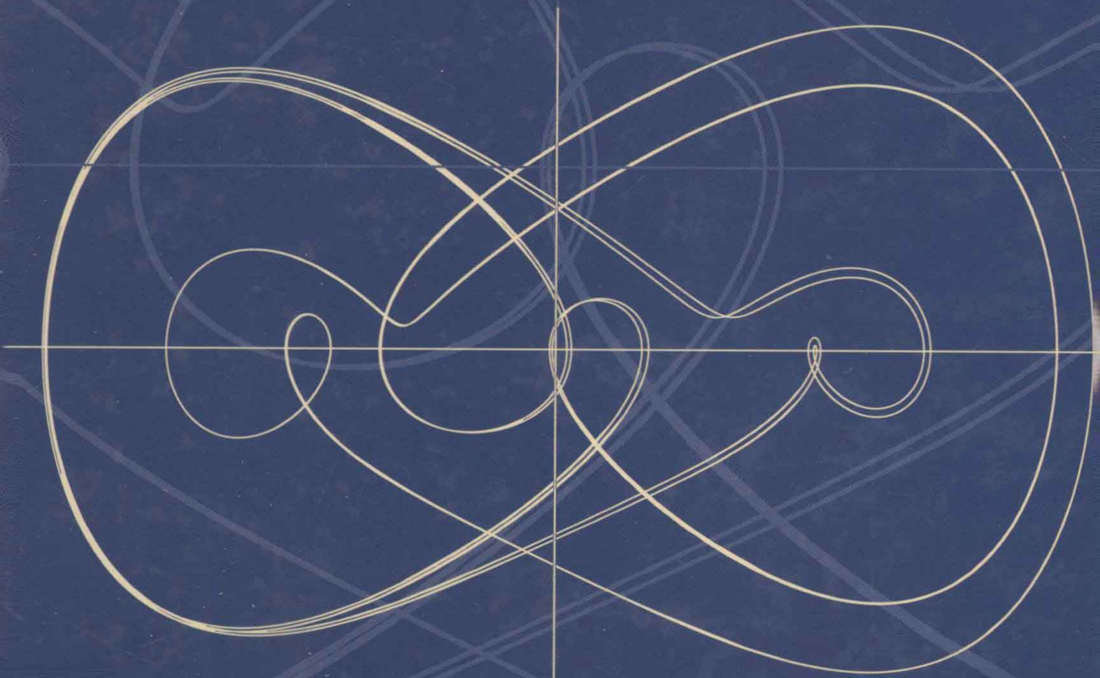
The Theory of DIFFERENTIAL EQUATIONS

Classical AND QUALITATIVE



KELLEY · PETERSON

The Theory of Differential Equations Classical and Qualitative

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The Theory of Differential Equations
Classical and Qualitative

We dedicate this book to our families:
Marilyn and Joyce
and
Tina, Carla, David, and Carrie.

Preface

Differential equations first appeared in the late seventeenth century in the work of Isaac Newton, Gottfried Wilhelm Leibniz, and the Bernoulli brothers, Jakob and Johann. They occurred as a natural consequence of the efforts of these great scientists to apply the new ideas of the calculus to certain problems in mechanics, such as the paths of motion of celestial bodies and the brachistochrone problem, which asks along which path from point P to point Q a frictionless object would descend in the least time. For over 300 years, differential equations have served as an essential tool for describing and analyzing problems in many scientific disciplines. Their importance has motivated generations of mathematicians and other scientists to develop methods of studying properties of their solutions, ranging from the early techniques of finding exact solutions in terms of elementary functions to modern methods of analytic and numerical approximation. Moreover, they have played a central role in the development of mathematics itself since questions about differential equations have spawned new areas of mathematics and advances in analysis, topology, algebra, and geometry have often offered new perspectives for differential equations.

This book provides an introduction to many of the important topics associated with ordinary differential equations. The material in the first six chapters is accessible to readers who are familiar with the basics of calculus, while some undergraduate analysis is needed for the more theoretical subjects covered in the final two chapters. The needed concepts from linear algebra are introduced with examples, as needed. Previous experience with differential equations is helpful but not required. Consequently, this book can be used either for a second course in ordinary differential equations or as an introductory course for well-prepared students.

The first chapter contains some basic concepts and solution methods that will be used throughout the book. Since the discussion is limited to first-order equations, the ideas can be presented in a geometrically simple setting. For example, dynamics for a first-order equation can be described in a one-dimensional space. Many essential topics make an appearance here: existence, uniqueness, intervals of existence, variation of parameters, equilibria, stability, phase space, and bifurcations. Since proofs of existence-uniqueness theorems tend to be quite technical, they are reserved for the last chapter.

Systems of linear equations are the major topic of the second chapter. An unusual feature is the use of the Putzer algorithm to provide a constructive method for solving linear systems with constant coefficients. The study of stability for linear systems serves as a foundation for nonlinear systems in the next chapter. The important case of linear systems with periodic coefficients (Floquet theory) is included in this chapter.

Chapter 3, on autonomous systems, is really the heart of the subject and the foundation for studying differential equations from a dynamical viewpoint. The discussion of phase plane diagrams for two-dimensional systems contains many useful geometric ideas. Stability of equilibria is investigated by both Liapunov's direct method and the method of linearization. The most important methods for studying limit cycles, the Poincare-Bendixson theorem and the Hopf bifurcation theorem, are included here. The chapter also contains a brief look at complicated behavior in three dimensions and at the use of *Mathematica* for graphing solutions of differential equations. We give proofs of many of the results to illustrate why these methods work, but the more intricate verifications have been omitted in order to keep the chapter to a reasonable length and level of difficulty.

Perturbation methods, which are among the most powerful techniques for finding approximations of solutions of differential equations, are introduced in Chapter 4. The discussion includes singular perturbation problems, an important topic that is usually not covered in undergraduate texts.

The next two chapters return to linear equations and present a rich mix of classical subjects, such as self-adjointness, disconjugacy, Green's functions, Riccati equations, and the calculus of variations.

Since many applications involve the values of a solution at different input values, boundary value problems are studied in Chapter 7. The contraction mapping theorem and continuity methods are used to examine issues of existence, uniqueness, and approximation of solutions of nonlinear boundary value problems.

The final chapter contains a thorough discussion of the theoretical ideas that provide a foundation for the subject of differential equations. Here we state and prove the classical theorems that answer the following questions about solutions of initial value problems: Under what conditions does a solution exist, is it unique, what type of domain does a solution have, and what changes occur in a solution if we vary the initial condition or the value of a parameter? This chapter is at a higher level than the first six chapters of the book.

There are many examples and exercises throughout the book. A significant number of these involve differential equations that arise in applications to physics, biology, chemistry, engineering, and other areas. To avoid lengthy digressions, we have derived these equations from basic principles only in the simplest cases.

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Contents

Chapter 1

First-Order Differential Equations

1.1 Basic Results

In the scientific investigation of any phenomenon, mathematical models are used to give quantitative descriptions and to derive numerical conclusions. These models can take many forms, and one of the most basic and useful is that of a differential equation, that is, an equation involving the rate of change of a quantity. For example, the rate of decrease of the mass of a radioactive substance, such as uranium, is known to be proportional to the present mass. If $m(t)$ represents the mass at time t, then we have that m satisfies the differential equation

$$m' = -km,$$

where k is a positive constant. This is an *ordinary differential equation* since it involves only the derivative of mass with respect to a single independent variable. Also, the equation is said to be of *first-order* because the highest order derivative appearing in the equation is first-order. An example of a second-order differential equation is given by Newton's second law of motion

$$mx'' = f(t, x, x'),$$

where m is the (constant) mass of an object moving along the x-axis and located at position $x(t)$ at time t, and $f(t, x(t), x'(t))$ is the force acting on the object at time t.

In this chapter, we will consider only first-order differential equations that can be written in the form

$$x' = f(t, x), \tag{1.1}$$

where $f : (a, b) \times (c, d) \to \mathbb{R}$ is continuous, $-\infty \leq a < b \leq \infty$, and $-\infty \leq c < d \leq \infty$.

Definition 1.1 We say that a function x is a *solution* of (1.1) on an interval $I \subset (a, b)$ provided $c < x(t) < d$ for $t \in I$, x is a continuously differentiable function on I, and

$$x'(t) = f(t, x(t)),$$

for $t \in I$.

Definition 1.2 Let $(t_0, x_0) \in (a, b) \times (c, d)$ and assume f is continuous on $(a, b) \times (c, d)$. We say that the function x is a solution of the initial value problem (IVP)

$$x' = f(t, x), \quad x(t_0) = x_0, \tag{1.2}$$

on an interval $I \subset (a, b)$ provided $t_0 \in I$, x is a solution of (1.1) on I, and

$$x(t_0) = x_0.$$

Note, for example, that if $(a, b) = (c, d) = (-\infty, \infty)$, then the function m defined by $m(t) = 400e^{-kt}$, $t \in (-\infty, \infty)$ is a solution of the IVP

$$m' = -km, \quad m(0) = 400$$

on the interval $I = (-\infty, \infty)$.

Solving an IVP can be visualized (see Figure 1) as finding a solution of the differential equation whose graph passes through the given point (t_0, x_0).

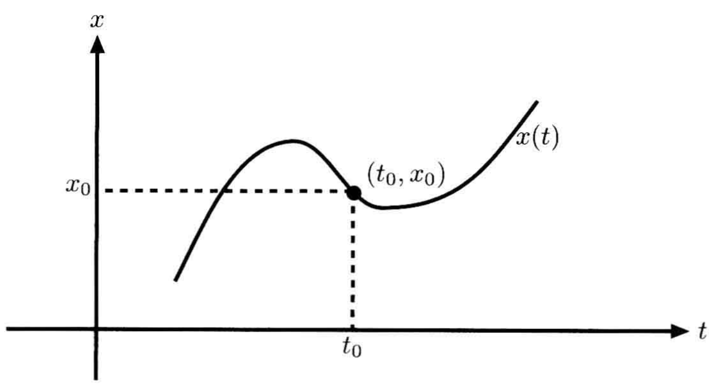

FIGURE 1. Graph of solution of IVP.

We state without proof the following important existence-uniqueness theorem for solutions of IVPs. Statements and proofs of some existence and uniqueness theorems will be given in Chapter 8.

Theorem 1.3 *Assume $f : (a, b) \times (c, d) \to \mathbb{R}$ is continuous, where $-\infty \leq a < b \leq \infty$ and $-\infty \leq c < d \leq \infty$. Let $(t_0, x_0) \in (a, b) \times (c, d)$, then the IVP (1.2) has a solution x with a maximal interval of existence $(\alpha, \omega) \subset (a, b)$, where $\alpha < t_0 < \omega$. If $a < \alpha$, then*

$$\lim_{t \to \alpha+} x(t) = c, \quad \text{or} \quad \lim_{t \to \alpha+} x(t) = d$$

and if $\omega < b$, then

$$\lim_{t \to \omega-} x(t) = c, \quad \text{or} \quad \lim_{t \to \omega-} x(t) = d.$$

If, in addition, the partial derivative of f with respect to x, f_x, is continuous on $(a, b) \times (c, d)$, then the preceding IVP has a unique solution.

We now give a couple of examples related to Theorem 1.3. The first example shows that if the hypothesis that the partial derivative f_x is continuous on $(a, b) \times (c, d)$ is not satisfied, then we might not have uniqueness of solutions of IVPs.

Example 1.4 (Nonuniqueness of Solutions to IVPs) If we drop an object from a bridge of height h at time $t = 0$ (assuming constant acceleration of gravity and negligible air resistance), then the height of the object after t units of time is $x(t) = -\frac{1}{2}gt^2 + h$. The velocity at time t is $x'(t) = -gt$, so by eliminating t, we are led to the IVP

$$x' = f(t, x) := -\sqrt{2g|h - x|}, \quad x(0) = h. \tag{1.3}$$

Note that this initial value problem has the constant solution $x(t) = h$, which corresponds to holding the object at bridge level without dropping it! We can find other solutions by separation of variables. If $h > x$, then

$$\int \frac{x'(t)\, dt}{\sqrt{2g(h - x(t))}} = -\int dt.$$

Computing the indefinite integrals and simplifying, we arrive at

$$x(t) = -\frac{g}{2}(t - C)^2 + h,$$

where C is an arbitrary constant. We can patch these solutions together with the constant solution to obtain for each $C > 0$

$$x(t) := \begin{cases} h, & \text{for} \quad t \le C \\ h - \frac{g}{2}(t - C)^2, & \text{for} \quad t > C. \end{cases}$$

Thus for each $C > 0$ we have a solution of the IVP (1.3) that corresponds to releasing the object at time C. Note that the function f defined by $f(t, x) = -\sqrt{2g|h - x|}$ is continuous on $(-\infty, \infty) \times (-\infty, \infty)$ so by Theorem 1.3 the IVP (1.3) has a solution, but f_x does not exist when $x = h$ so we cannot use Theorem 1.3 to get that the IVP (1.3) has a unique solution. △

To see how bad nonuniqueness of solutions of initial value problems can be, we remark that in Hartman [**17**], pages 18–23, an example is given of a scalar equation $x' = f(t, x)$, where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, is continuous, where for every IVP (1.2) there is more than one solution on $[t_0, t_0 + \epsilon]$ and $[t_0 - \epsilon, t_0]$ for arbitrary $\epsilon > 0$.

The next example shows even if the hypotheses of Theorem 1.3 hold the solution of the IVP might only exist on a proper subinterval of (a, b).

Example 1.5 Let k be any nonzero constant. The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(t, x) = kx^2$ is continuous and has a continuous partial derivative with respect to x. By Theorem 1.3, the IVP

$$x' = kx^2, \quad x(0) = 1$$

has a unique solution with a maximal interval of existence (α, ω). Using separation of variables, as in the preceding example, we find

$$x(t) = \frac{1}{C - kt}.$$

When we apply the initial condition $x(0) = 1$, we have $C = 1$, so that the solution of the IVP is

$$x(t) = \frac{1}{1 - kt},$$

with maximal interval of existence $(-\infty, 1/k)$ if $k > 0$ and $(1/k, \infty)$ if $k < 0$. In either case, $x(t)$ goes to infinity as t approaches $1/k$ from the appropriate direction.

Observe the implications of this calculation in case $x(t)$ is the density of some population at time t. If $k > 0$, then the density of the population is growing, and we conclude that growth cannot be sustained at a rate proportional to the square of density because the density would have to become infinite in finite time! On the other hand, if $k < 0$, the density is declining, and it is theoretically possible for the decrease to occur at a rate proportional to the square of the density, since $x(t)$ is defined for all $t > 0$ in this case. Note that $\lim_{t\to\infty} x(t) = 0$ if $k < 0$. △

1.2 First-Order Linear Equations

An important special case of a first-order differential equation is the first-order linear differential equation given by

$$x' = p(t)x + q(t), \tag{1.4}$$

where we assume that $p : (a, b) \to \mathbb{R}$ and $q : (a, b) \to \mathbb{R}$ are continuous functions, where $-\infty \le a < b \le \infty$. In Chapter 2, we will study systems of linear equations involving multiple unknown functions. The next theorem shows that a single linear equation can always be solved in terms of integrals.

Theorem 1.6 (Variation of Constants Formula) *If $p : (a, b) \to \mathbb{R}$ and $q : (a, b) \to \mathbb{R}$ are continuous functions, where $-\infty \le a < b \le \infty$, then the unique solution x of the IVP*

$$x' = p(t)x + q(t), \quad x(t_0) = x_0, \tag{1.5}$$

where $t_0 \in (a, b)$, $x_0 \in \mathbb{R}$, is given by

$$x(t) = e^{\int_{t_0}^t p(\tau)\, d\tau} x_0 + e^{\int_{t_0}^t p(\tau)\, d\tau} \int_{t_0}^t e^{-\int_{t_0}^s p(\tau)\, d\tau} q(s)\, ds,$$

$t \in (a, b)$.

Proof Here the function f defined by $f(t,x) = p(t)x + q(t)$ is continuous on $(a,b) \times (-\infty,\infty)$ and $f_x(t,x) = p(t)$ is continuous on $(a,b) \times (-\infty,\infty)$. Hence by Theorem 1.3 the IVP (1.5) has a unique solution with a maximal interval of existence $(\alpha,\omega) \subset (a,b)$ [the existence and uniqueness of the solution of the IVP (1.5) and the fact that this solution exists on the whole interval (a,b) follows from Theorem 8.65]. Let

$$x(t) := e^{\int_{t_0}^t p(\tau)\,d\tau} x_0 + e^{\int_{t_0}^t p(\tau)\,d\tau} \int_{t_0}^t e^{-\int_{t_0}^s p(\tau)\,d\tau} q(s)\,ds$$

for $t \in (a,b)$. We now show that x is the solution of the IVP (1.5) on the whole interval (a,b). First note that $x(t_0) = x_0$ as desired. Also,

$$\begin{aligned} x'(t) &= p(t)e^{\int_{t_0}^t p(\tau)\,d\tau} x_0 + p(t)e^{\int_{t_0}^t p(\tau)\,d\tau} \int_{t_0}^t e^{-\int_{t_0}^s p(\tau)d\tau} q(s)\,ds + q(t) \\ &= p(t)\left[e^{\int_{t_0}^t p(\tau)\,d\tau} x_0 + e^{\int_{t_0}^t p(\tau)\,d\tau} \int_{t_0}^t e^{-\int_{t_0}^s p(\tau)d\tau} q(s)\,ds\right] + q(t) \\ &= p(t)x(t) + q(t) \end{aligned}$$

for $t \in (a,b)$. □

In Theorem 2.40, we generalize Theorem 1.6 to the vector case. We now give an application of Theorem 1.6.

Example 1.7 (Newton's Law of Cooling) Newton's law of cooling states that the rate of change of the temperature of an object is proportional to the difference between its temperature and the temperature of the surrounding medium. Suppose that the object has an initial temperature of 40 degrees. If the temperature of the surrounding medium is $70 + 20e^{-2t}$ degrees after t minutes and the constant of proportionality is $k = -2$, then the initial value problem for the temperature $x(t)$ of the object at time t is

$$x' = -2(x - 70 - 20e^{-2t}), \quad x(0) = 40.$$

By the variation of constants formula, the temperature of the object after t minutes is

$$\begin{aligned} x(t) &= 40e^{\int_0^t -2d\tau} + e^{\int_0^t -2d\tau} \int_0^t e^{\int_0^s 2d\tau}(140 + 40e^{-2s})\,ds \\ &= 40e^{-2t} + e^{-2t} \int_0^t \left(140e^{2s} + 40\right)\,ds \\ &= 40e^{-2t} + e^{-2t}[70(e^{2t} - 1) + 40t] \\ &= 10(4t - 3)e^{-2t} + 70. \end{aligned}$$

Sketch the graph of x. Does the temperature of the object exceed 70 degrees at any time t? △

1.3 Phase Line Diagrams

If, in equation (1.1), f depends only on x, we get the autonomous differential equation

$$x' = f(x). \tag{1.6}$$

We always assume $f : \mathbb{R} \to \mathbb{R}$ is continuous and usually we assume its derivative is also continuous. The fundamental property of autonomous differential equations is that translating any solution of the autonomous differential equation along the t-axis produces another solution.

Theorem 1.8 *If x is a solution of the autonomous differential equation (1.6) on an interval (a, b), where $-\infty \le a < b \le \infty$, then for any constant c, the function y defined by $y(t) := x(t-c)$, for $t \in (a+c, b+c)$ is a solution of (1.6) on $(a+c, b+c)$.*

Proof Assume x is a solution of the autonomous differential equation (1.6) on (a, b); then x is continuously differentiable on (a, b) and

$$x'(t) = f(x(t)),$$

for $t \in (a, b)$. Replacing t by $t - c$ in this last equation, we get that

$$x'(t-c) = f(x(t-c)),$$

for $t \in (a+c, b+c)$. By the chain rule of differentiation we get that

$$\frac{d}{dt}[x(t-c)] = f(x(t-c)),$$

for $t \in (a+c, b+c)$. Hence if $y(t) := x(t-c)$ for $t \in (a+c, b+c)$, then y is continuously differentiable on $(a+c, b+c)$ and we get the desired result that

$$y'(t) = f(y(t)),$$

for $t \in (a+c, b+c)$. □

Definition 1.9 If $f(x_0) = 0$ we say that x_0 is an *equilibrium point* for the differential equation (1.6). If, in addition, there is a $\delta > 0$ such that $f(x) \neq 0$ for $|x - x_0| < \delta$, $x \neq x_0$, then we say x_0 is an *isolated equilibrium point.*

Note that if x_0 is an equilibrium point for the differential equation (1.6), then the constant function $x(t) = x_0$ for $t \in \mathbb{R}$ is a solution of (1.6) on $\mathbb{R}$.

Example 1.10 (Newton's Law of Cooling) Consider again Newton's law of cooling as in Example 1.7, where in this case the temperature of the surrounding medium is a constant 70 degrees. Then we have that the temperature $x(t)$ of the object at time t satisfies the differential equation

$$x' = -2(x - 70).$$

Note that $x = 70$ is the only equilibrium point. All solutions can be written in the form

$$x(t) = De^{-2t} + 70,$$

where D is an arbitrary constant. If we translate a solution by a constant amount c along the t-axis, then

$$x(t-c) = De^{-2(t-c)} + 70 = De^{2c}e^{-2t} + 70$$

is also a solution, as predicted by Theorem 1.8. Notice that if the temperature of the object is initially greater than 70 degrees, then the temperature will decrease and approach the equilibrium temperature 70 degrees as t goes to infinity. Temperatures starting below 70 degrees will increase toward the limiting value of 70 degrees. A simple graphical representation of this behavior is a "phase line diagram," (see Figure 2) showing the equilibrium point and the direction of motion of the other solutions.

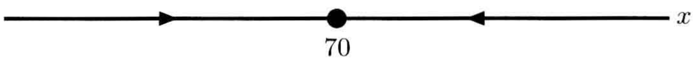

FIGURE 2. Phase line diagram of $x' = -2(x - 70)$.

△

Definition 1.11 Let ϕ be a solution of (1.6) with maximal interval of existence (α, ω). Then the set

$$\{\phi(t) : t \in (\alpha, \omega)\}$$

is called an *orbit* for the differential equation (1.6).

Note that the orbits for

$$x' = -2(x - 70)$$

are the sets

$$(-\infty, 70), \quad \{70\}, \quad (70, \infty).$$

A convenient way of thinking about phase line diagrams is to consider $x(t)$ to be the position of a point mass moving along the x-axis and $x'(t) = f(x(t))$ to be its velocity. The phase line diagram then gives the direction of motion (as determined by the sign of the velocity). An orbit is just the set of all locations of a continuous motion.

Theorem 1.12 *Assume that $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then two orbits of (1.6) are either disjoint sets or are the same set.*

Proof Let ϕ_1 and ϕ_2 be solutions of (1.6). We will show that if there are points t_1, t_2 such that

$$\phi_1(t_1) = \phi_2(t_2),$$