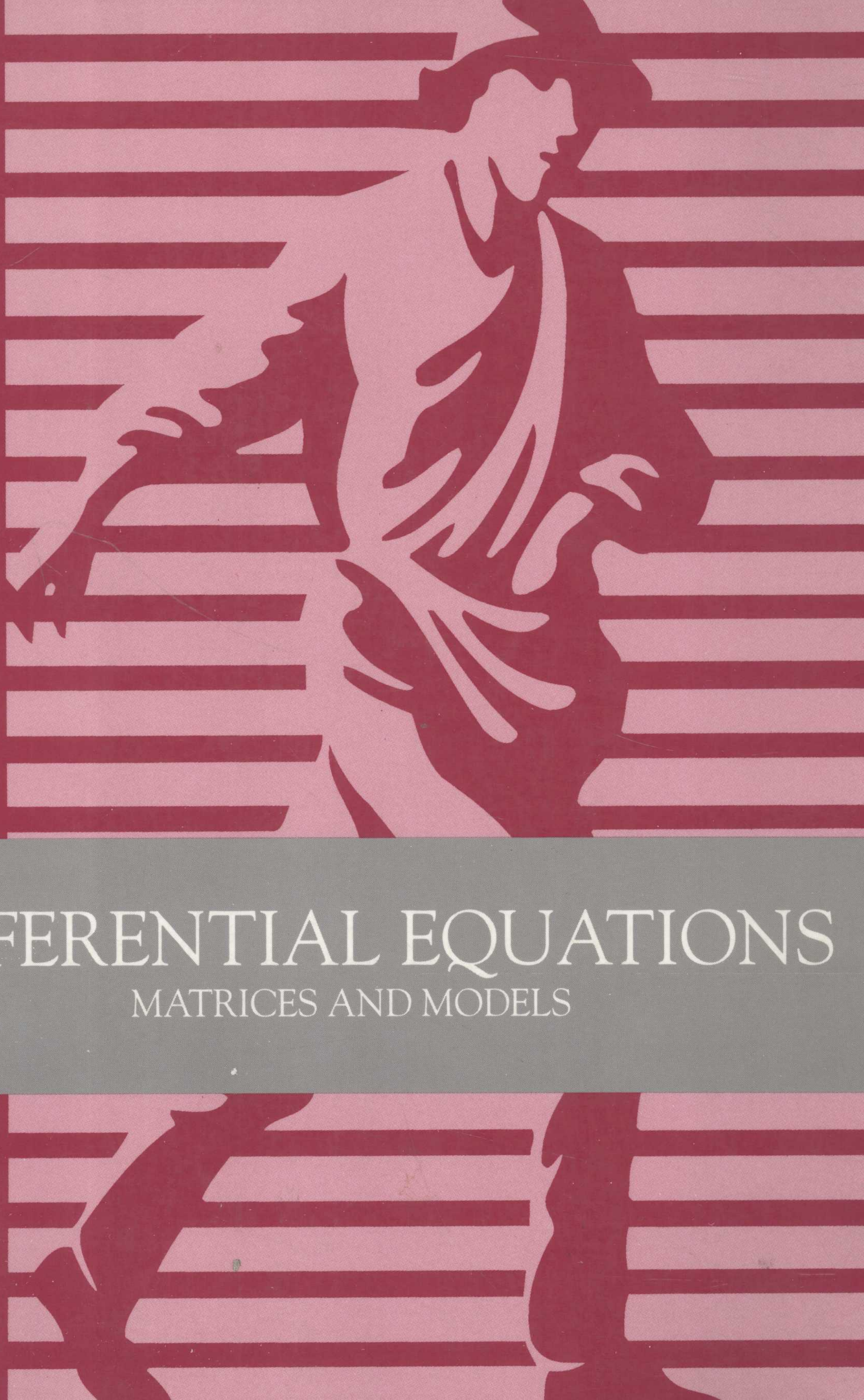
PRENTICE
HALL
INTERNATIONAL
EDITIONS



DIFFERENTIAL EQUATIONS

MATRICES AND MODELS

PAUL BUGL

DIFFERENTIAL EQUATIONS
Matrices and Models

PAUL BUGL
University of Hartford

PRENTICE HALL INTERNATIONAL, INC.
Englewood Cliffs, New Jersey 07632

A Simon & Schuster Company
Englewood Cliffs, New Jersey 07632

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

ISBN 0-13-320995-4

Prentice Hall International (UK) Limited, *London*
Prentice Hall of Australia Pty. Limited, *Sydney*
Prentice Hall Canada Inc., *Toronto*
Prentice Hall Hispanoamericana, S.A., *Mexico*
Prentice Hall of India Private Limited, *New Delhi*
Prentice Hall of Japan, Inc., *Tokyo*
Simon & Schuster Asia Pte. Ltd., *Singapore*
Editora Prentice Hall do Brasil, Ltda., *Rio de Janeiro*
Prentice-Hall, Inc. Englewood Cliffs, New Jersey

DIFFERENTIAL EQUATIONS

PREFACE

Historically, the study of differential equations was a major driving force in the development of the ideas of the calculus. Models of physical problems were formulated in terms of differential equations whose solution often gave rise to new areas of study in both mathematics and the sciences. Problems worthy of study have become ever more difficult for the classical techniques. Reasonably accurate models frequently require the use of several dependent variables and a utilization of the power of matrix algebra.

Previously, the study of differential equations was a collection of sometimes unrelated techniques. However, change is moving through the course. The use of computer algebra systems renders superfluous some of the elaborate techniques needed for special cases. More emphasis on systems of linear ordinary differential equations reflects greater realism and requires an early introduction to linear algebra. As material is removed from the course, more attention can and should be given to modeling; students must learn how to properly formulate the problems that they, in turn, should solve.

This book is intended to provide a modern study of differential equations in the spirit of reasonable, but overdue, reform. Matrix algebra is presented along with many of the elementary numerical techniques needed for the computer implementation of its procedures. The methods for the solution of differential equations appear only after models which generate the equations have been studied. The ultimate goal of the text is the use of matrix methods for the solution of systems of linear ordinary differential equations. The reader who comes away with an understanding of the power of the state-transition matrix approach will have achieved the aim of the book.

A large proportion of students taking a course in differential equations major in either engineering or the physical sciences. Such students will have taken two semesters of calculus-based physics and possibly both statics and dynamics and/or a first course in electrical circuits. For this reason, the models most frequently used are those of classical mechanics and network analysis. They have seen these before and are usually more comfortable with them. Any student who sees these models in another context will suffer no harm and repeated study can only reinforce concepts learned elsewhere. To make it easier for students, the Lagrangian approach is used for formulating mechanics problems, thus avoiding the difficulty of constructing the free body diagram.

Many of the classical solution methods are not presented in this book. Exact equations and other such are nowhere to be found. A method appears only if it is useful in solving equations arising from reasonable models. Undetermined coefficients is given short shrift. Some methods which seem not to be a common part of the curriculum have been included, e.g., the LU-decomposition, pivoting and scaling, ill-conditioned systems, boundary and initial value Green's functions, the matrix exponential, controllability, and observability. On the other hand, series solutions are discussed in detail because none of the usual computer algebra systems can generate the general form of a solution for a simple but arbitrary second order equation. It is also possible to use this book for a more traditional course, if that it the aim.

The matrix language Matlab is used as an adjunct to keep one from being overwhelmed by the details involved in programming languages like Pascal, C, or FORTRAN. Matlab's notation is rapidly becoming universally adopted. The use of a computer algebra system would be helpful, insofar as large parts of several chapters could be omitted. In that case, use could be made of the appropriate lab manual which accompanies this text.

An attempt has been made to provide the reader with a large number of detailed nontrivial examples to enhance self-study.

Some say a book is only as good as its problems. The exercise sets have been designed to be broad-based. There are some simple computational problems, verifications, many "thought" problems, and a large number of modeling applications. Each chapter ends with a set of supplementary and complementary problems which are designed to extend either the material in the text or the reader. These problems are meant to be more interesting.

Outline

The first two chapters can be covered in either order.

Sections are organized logically so as to completely cover a single subject area. For this reason, one section can rarely be covered in one fifty minute class. Instead, subsections may be the units of class time.

Chapter 1 covers the lion's share of complex matrix algebra. Systems of linear algebraic equations are solved using pivotal reduction (Gauss-Jordan elimination with partial pivoting). The ideas of leading and nonleading variables and their relation to the rank and nullity of the coefficient matrix and its row-reduced echelon form are emphasized. Inverses appear as theoretical constructs for the further study of linear equations. Determinants are defined in terms of elementary row operations and applied to solving systems of linear equations and finding inverses of square matrices. Computer solution of linear systems and the LU-decomposition are discussed as optional material.

Chapter 2 gives an introduction to models and differential equations. Graphical and numerical solutions are discussed before any analytical methods. First order equations are solved using either separation of variables or the integrating factor for linear equations. As an option, Runge-Kutta methods are derived as multistage methods.

Chapter 3 introduces the complementary concepts of linear spaces and linear transformations. Infinite dimensional spaces are not shunned. Proofs are given when they use the ideas of systems of linear equations or provide a method of solution. Many results are only stated. Induced matrix norms and condition numbers are discussed and used as an assessment of the computational solvability of a linear system. Optional sections on the Gram-Schmidt procedure and fundamental subspaces associated with a matrix rounds out the chapter.

Chapter 4 introduces linear ODE's in terms of differential operators and trial solutions. Some of the ideas of linear spaces are used. For the most part, the treatment is fairly classical, but the intention is that computer software be used to speed the presentation. Applications to mechanical and electrical systems are given. Boundary value Green's functions are introduced as an optional technique for reducing a nonhomogeneous problem to a quadrature. Most of this material could be omitted if the course will utilize a computer algebra system.

Chapter 5 develops the ideas of the Laplace transform in six relatively compact sections. The evaluation of the transform and the inverse transform are unified into a single section for each. The geometrical interpretation of the convolution product is given. Transforms are then applied to solving linear ODE's and finding Green's functions. Applications to elastic beam problems illustrate the solution of problems with discontinuous forcing functions. A computer algebra system would be a valuable tool to eliminate much of the algebraic tedium involved in many of the calculations.

Chapter 6 begins with models which require more than one dependent variable, after which systems of equations are reduced to standard matrix form. After the ideas of linear systems are developed, the matrix eigenvalue problem is studied in some detail, including the Jordan normal form. Linear systems solution methods are then presented. The state-transition matrix is introduced and functions of matrices are discussed. Several methods for the calculation of the matrix exponential are given and applied to solving homogeneous and nonhomogeneous linear systems of ODE's. Phase portraits of linear systems are studied in terms of the eigenvalues of the coefficient matrix. Initial value Green's functions and the ideas of controllability and observability of a system are also discussed, as are Laplace transform methods for linear systems.

Chapter 7 provides an overview of nonlinear systems. In addition to the usual discussion of phase plane analysis, stability, and limit cycles, there is an outline of regular perturbation methods. Brief introductions to singular perturbations and chaotic systems are also given.

Chapter 8 contains a fairly detailed coverage of series solutions to linear equations with variable coefficients including the method of Frobenius.

Chapter 9 closes things out with an optional survey of special functions.

A listing of the Matlab commands that a student might use are collected in the first appendix. Other appendices are included as reviews or brief discussions of complex numbers, complex functions, unit steps and impulses, partial fraction expansions, and infinite series.

Answers are provided to most of the odd-numbered problems.

Ancillary Material

In addition to the text, other materials are available for use in teaching a course from this book.

- Student version of Matlab 4.0 which includes the Maple kernel can be packaged with the text. This includes the valuable command reference book. There are Windows and Macintosh versions.

- A student solution manual contains the fairly complete solutions to most of the odd-numbered problems in the exercise sets in the book.

- An instructor's solutions manual contains solutions to almost all of the problems in the book, including the supplementary and complementary problems.

- A computer lab manual which provides the student with problems and projects for exploration and experimentation. It comes in three separate versions: Matlab 4.0, Maple, and Mathematica.

Acknowledgments

What seems like countless students have worked through various drafts of the manuscript. Their spirits did not appear to be dampened by what must surely be a law of Nature, the more you proof read anything, the more errors creep in. They found the errors and made many suggestions about the exposition and organization. Special thanks go to Mark Linnick who generated the figures and graphs and also to Jean-Marie Rennetaud whose eagle eye caught many errors in preliminary drafts.

Reviewers have had a pronounced effect on the final outcome. Their suggestions were extremely valuable and, with very few exceptions, included in the text. I wish to thank the following reviewers:

Terry Herdman, Virginia Polytechnic Institute

Dar-Vieg Ho, Georgia Institute of Technology

Karl Kosler, University of Wisconsin

Calvin Piston, John Brown University

Lawrence Runyan, Shoreline Community College

Michael Simon, Housatonic Community College

My thanks go to my latest editor George Lobell, whose constant prodding has resulted in a timely publication schedule. Amy Hendrickson of TEXnology Inc. deserves credit for the speedy composition and visually pleasing design of this book.

Last, and most surely not least, my wife Dae deserves a debt of gratitude that goes beyond what mere words can express.

Any errors are mine, and mine alone. All suggestions, comments, and criticisms are welcome and will be gratefully received. You can write (using o-mail) to me at the Department of Math/Physics/CS at the University of Hartford, West Hartford, CT 06117-1500 or use my (inordinately long) e-mail address:

`bugl%uhavax.dnet@ipgate.hartford.edu`,

although I have been known to receive messages sent to `bugl@hartford.edu`.

CONTENTS

1

MATRIX ALGEBRA

1.1 PREVIEW

Matrices lie at the very foundation of applied mathematics. They are the bedrock on which many models are built. Whenever more than one dependent variable is involved, a problem can be phrased in the language of matrix theory so that a number of powerful methods can be brought to bear.

We begin with some linear multivariate models that can be simplified by the use of matrices. All of the necessary operations of matrix algebra and their properties are introduced. Special matrices are defined and studied. Most problems that can be formulated in the language of matrices eventually lead to the problem of solving a system of linear algebraic equations. Section 1.4 presents the method of pivotal reduction for arriving at a solution when one exists. It also discusses the forms and existence of solutions.

The inverse of a matrix is introduced as a formal alternative to the use of the method of pivotal reduction. We restrict our attention to square matrices and their reduction by elementary matrices. Some of the implications of invertibility are explored. Finally, a model of cascaded two-port networks is introduced and the inverse is used to construct solutions.

The concept of the determinant of a square matrix as an alternating multilinear functional is developed by way of elementary row operations. The expansion by cofactors is stated, as is Cramer's Rule for solving square systems. Then the adjugate matrix is used to compute the inverse of a nonsingular matrix.

Two optional sections deal with the computer implementation of techniques for solving systems of linear equations. The method of Gaussian elimination is compared with pivotal reduction with a discussion of pivoting and scaling. A simple example of an ill-conditioned system is given, and the effects of computer round-off are discussed. The methodology of the LU-decomposition is presented and applied to solving linear systems.

1.2 LINEAR MULTIVARIATE MODELS

Mathematical models often devolve upon solving systems of simultaneous equations. Frequently, these equations involve the unknowns raised to the first power and there are no products of unknowns. Such models pervade applied mathematics. What follows is but a small sample of them.

1.2.1 Steady State AC Electrical Circuits

An electrical **circuit** is a collection of resistors, inductors, and capacitors connected by idealized wires joined at **nodes**. When there is an alternating current source, each of the circuit elements can be thought of as having an **impedance**, Z, that satisfies the generalized **Ohm's Law**,

$$E = IZ,$$

where E is the voltage drop across the element and I is the current flowing through it. In a general alternating current circuit, the impedance Z is complex and has the form

$$Z = R + i\left(\omega L - \frac{1}{\omega C}\right),$$

where R is the resistance (measured in ohms Ω), L is the inductance (measured in henrys H), C is the capacitance (measured in farads F), and $f = \omega/2\pi$ is the frequency (measured in hertz Hz) of the AC source.

Most circuits are composed of several loops, each of which has several elements on it. One method of analyzing a circuit is to assign currents to each loop, write Ohm's Law separately for each element, and use the **Kirchhoff Voltage Law**, which says that the sum of the voltage drops around a loop must be zero.

This procedure has the advantage of always resulting in as many equations as there are unknowns.

▪ **EXAMPLE 1.1** The circuit in Figure 1.1 has five loops with unknown loop currents I_1, I_2, I_3, I_4, and I_5 drawn clockwise. If the voltage E is known, set up the linear equations that determine these currents.

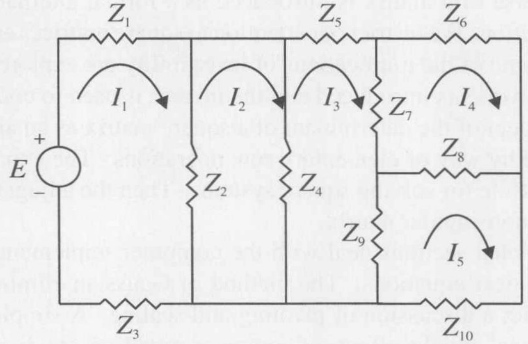


Fig. 1.1