THE CALCULUS WITH ANALYTIC GEOMETRY
third edition

Part II Infinite series, vectors, and functions of several variables

Louis Leithold
UNIVERSITY OF SOUTHERN CALIFORNIA

HARPER & ROW, PUBLISHERS
New York, Evanston, San Francisco, London

To Gordon Marc

Sponsoring Editor: George J. Telecki
Project Editor: Karen A. Judd
Designer: Rita Naughton
Production Supervisor: Francis X. Giordano
Compositor: Progressive Typographers
Printer and Binder: Kingsport Press
Art Studio: J & R Technical Services Inc.
Cover and chapter opening art: "Study Light" by Patrick Caulfield

THE CALCULUS WITH ANALYTIC GEOMETRY, Third Edition
Part II: Infinite Series, Vectors, and Functions of Several Variables.

Library of Congress Cataloging in Publication Data
Leithold, Louis.
The calculus, with analytic geometry.

Includes index.
CONTENTS: pt. 1. Functions of one variable, plane analytic geometry, and infinite series.—pt. 2. Infinite series, vectors, and functions of several variables.
1. Calculus. 2. Geometry, Analytic. I. Title.
QA303.L428 1976 515'.15 75-25981
ISBN 0-06-043949-1 (v. 2)

Preface

This third edition of THE CALCULUS WITH ANALYTIC GEOMETRY, like the other two, is designed for prospective mathematics majors as well as for students whose primary interest is in engineering, the physical sciences, or nontechnical fields. A knowledge of high-school algebra and geometry is assumed.

The text is available either in one volume or in two parts: Part I consists of the first sixteen chapters, and Part II comprises Chapters 16 through 21 (Chapter 16 on Infinite Series is included in both parts to make the use of the two-volume set more flexible). The material in Part I consists of the differential and integral calculus of functions of a single variable and plane analytic geometry, and it may be covered in a one-year course of nine or ten semester hours or twelve quarter hours. The second part is suitable for a course consisting of five or six semester hours or eight quarter hours. It includes the calculus of several variables and a treatment of vectors in the plane, as well as in three dimensions, with a vector approach to solid analytic geometry.

The objectives of the previous editions have been maintained. I have endeavored to achieve a healthy balance between the presentation of elementary calculus from a rigorous approach and that from the older, intuitive, and computational point of view. Bearing in mind that a textbook should be written for the student, I have attempted to keep the presentation geared to a beginner's experience and maturity and to leave no step unexplained or omitted. I desire that the reader be aware that proofs of theorems are necessary and that these proofs be well motivated and carefully explained so that they are understandable to the student who has achieved an average mastery of the preceding sections of the book. If a theorem is stated without proof, I have generally augmented the discussion by both figures and examples, and in such cases I have always stressed that what is presented is an illustration of the content of the theorem and is not a proof.

Changes in the third edition occur in the first five chapters. The first section of Chapter 1 has been rewritten to give a more detailed exposition of the real-number system. The introduction to analytic geometry in this chapter includes the traditional material on straight lines as well as that of the circle, but a discussion of the parabola is postponed to Chapter 14, The Conic Sections. Functions are now introduced in Chapter 1. I have defined a function as a set of ordered pairs and have used this idea to point up the concept of a function as a correspondence between sets of real numbers.

The treatment of limits and continuity which formerly consisted of ten sections in Chapter 2 is now in two chapters (2 and 4), with the chapter on the derivative placed between them. The concepts of limit and continuity are at the heart of any first course in the calculus. The notion of a limit of a function is first given a step-by-step motivation, which brings the discussion from computing the value of a function near a number, through an intuitive treatment of the limiting process, up to a rigorous epsilon-delta definition. A sequence of examples progressively graded in difficulty is included. All the limit theorems are stated, and some proofs are presented in the text, while other proofs have been outlined in the exercises. In the discussion of continuity, I have used as examples and counterexamples "common, everyday" functions and have avoided those that would have little intuitive meaning.

In Chapter 3, before giving the formal definition of a derivative, I have defined the tangent line to a curve and instantaneous velocity in rectilinear motion in order to demonstrate in advance that the concept of a derivative is of wide application, both geometrical and physical. Theorems on differentiation are proved and illustrated by examples. Application of the derivative to related rates is included.

Additional topics on limits and continuity are given in Chapter 4. Continuity on a closed interval is defined and discussed, followed by the introduction of the Extreme-Value Theorem, which involves such functions. Then the Extreme-Value Theorem is used to find the absolute extrema of functions continuous on a closed interval. Chapter 4 concludes with Rolle's Theorem and the Mean-Value Theorem. Chapter 5 gives additional applications of the derivative, including problems on curve sketching as well as some related to business and economics.

The antiderivative is treated in Chapter 6. I use the term "antidifferentiation" instead of indefinite integration, but the standard notation $\int f(x)\, dx$ is retained so that you are not given a bizarre new notation that would make the reading of standard references difficult. This notation will suggest that some relation must exist between definite integrals, introduced in Chapter 7, and antiderivatives, but I see no harm in this as long as the presentation gives the theoretically proper view of the definite integral as the limit of sums. Exercises involving the evaluation of definite integrals by finding limits of sums are given in Chapter 7 to stress that this is how they are calculated. The introduction of the definite integral follows the definition of the measure of the area under a curve as a limit of sums. Elementary properties of the definite integral are derived and the fundamental theorem of the calculus is proved. It is emphasized that this is a theorem, and an important one, because it provides us with an alternative to computing limits of sums. It is also emphasized that the definite integral is in no sense some special type of antiderivative. In Chapter 8 I have given numerous applications of definite integrals. The presentation highlights not only the manipulative techniques but also the fundamental principles involved. In each application, the definitions of the new terms are intuitively motivated and explained.

The treatment of logarithmic and exponential functions in Chapter 9 is the modern approach. The natural logarithm is defined as an integral, and after the discussion of the inverse of a function, the exponential function is defined as the inverse of the natural logarithmic function. An irrational power of a real number is then defined. The trigonometric functions are defined in Chapter 10 as functions assigning numbers to numbers. The important trigonometric identities are derived and used to obtain the formulas for the derivatives and integrals of these functions. Following are sections on the differentiation and integration of the trigonometric functions as well as of the inverse trigonometric functions.

Chapter 11, on techniques of integration, involves one of the most important computational aspects of the calculus. I have explained the theoretical backgrounds of each different method after an introductory motivation. The mastery of integration techniques depends upon the examples, and I have used as illustrations problems that the student will certainly meet in practice, those which require patience and persistence to solve. The material on the approximation of definite integrals includes the statement of theorems for computing the bounds of the error involved in these approximations. The theorems and the problems that go with them, being self-contained, can be omitted from a course if the instructor so wishes.

A self-contained treatment of hyperbolic functions is in Chapter 12. This chapter may be studied immediately following the discussion of the circular trigonometric functions in Chapter 10, if so desired. The geometric interpretation of the hyperbolic functions is postponed until Chapter 17 because it involves the use of parametric equations.

Polar coordinates and some of their applications are given in Chapter 13. In Chapter 14, conics are treated as a unified subject to stress their natural and close relationship to each other. The parabola is discussed in the first two sections. Then equations of the conics in polar coordinates are treated, and the cartesian equations of the ellipse and the hyperbola are derived from the polar equations. The topics of indeterminate forms, improper integrals, and Taylor's formula, and the computational techniques involved, are presented in Chapter 15.

I have attempted in Chapter 16 to give as complete a treatment of infinite series as is feasible in an elementary calculus text. In addition to the customary computational material, I have included the proof of the equivalence of convergence and boundedness of monotonic sequences based on the completeness property of the real numbers and the proofs of the computational processes involving differentiation and integration of power series.

The first five sections of Chapter 17 on vectors in the plane can be taken up after Chapter 5 if it is desired to introduce vectors earlier in the course. The approach to vectors is modern, and it serves both as an introduction to the viewpoint of linear algebra and to that of classical vector analysis. The applications are to physics and geometry. Chapter 18 treats vectors in three-dimensional space, and, if desired, the topics in the first three sections of this chapter may be studied concurrently with the corresponding topics in Chapter 17.

Limits, continuity, and differentiation of functions of several variables are considered in Chapter 19. The discussion and examples are applied mainly to functions of two and three variables; however, statements of most of the definitions and theorems are extended to functions of n variables.

In Chapter 20, a section on directional derivatives and gradients is followed by a section that shows the application of the gradient to finding an equation of the tangent plane to a surface. Applications of partial derivatives to the solution of extrema problems and an introduction to Lagrange multipliers are presented, as well as a section on applications of partial derivatives in economics. Three sections, new in the third edition, are devoted to line integrals and related topics. The double integral of a function of two variables and the triple integral of a function of three variables, along with some applications to physics, engineering, and geometry, are given in Chapter 21.

New to this edition is a short table of integrals appearing on the front and back endpapers. However, as stated in Chapter 11, you are advised to use a table of integrals only after you have mastered integration.

Louis Leithold

ACKNOWLEDGMENTS

Reviewers of *The Calculus with Analytic Geometry*

Professor William D. Bandes, San Diego Mesa College
Professor Archie D. Brock, East Texas State University
Professor Phillip Clarke, Los Angeles Valley College
Professor Reuben W. Farley, Virginia Commonwealth University
Professor Jacob Golightly, Jacksonville University
Professor Robert K. Goodrich, University of Colorado
Professor Albert Herr, Drexel University
Professor James F. Hurley, University of Connecticut
Professor Gordon L. Miller, Wisconsin State University
Professor William W. Mitchell, Jr., Phoenix College
Professor Roger B. Nelsen, Lewis and Clark College
Professor Robert A. Nowlan, Southern Connecticut State College
Sister Madeleine Rose, Holy Names College
Professor George W. Schultz, St. Petersburg Junior College
Professor Donald R. Sherbert, University of Illinois
Professor John Vadney, Fulton-Montgomery Community College
Professor David Whitman, San Diego State College

Production Staff at Harper & Row

George Telecki, Mathematics Editor
Karen Judd, Project Editor
Rita Naughton, Designer

Assistants for Answers to Exercises

Jacqueline Dewar, Loyola Marymount University
Ken Kast, Logicon, Inc.
Jean Kilmer, West Covina Unified School District

Cover and Chapter Opening Artist

Patrick Caulfield, London, England

To these people and to all the users of the first and second editions who have suggested changes, I express my deep appreciation.

L. L.

Contents

Preface *vii*

Chapter 16
INFINITE SERIES
page 659

16.1 Sequences *660*
16.2 Monotonic and Bounded Sequences *667*
16.3 Infinite Series of Constant Terms *673*
16.4 Infinite Series of Positive Terms *684*
16.5 The Integral Test *694*
16.6 Infinite Series of Positive and Negative Terms *697*
16.7 Power Series *707*
16.8 Differentiation of Power Series *713*
16.9 Integration of Power Series *722*
16.10 Taylor Series *729*
16.11 The Binomial Series 738

Chapter 17
VECTORS IN THE PLANE AND PARAMETRIC EQUATIONS
page 745

17.1 Vectors in the Plane *746*
17.2 Properties of Vector Addition and Scalar Multiplication *751*
17.3 Dot Product *756*
17.4 Vector-Valued Functions and Parametric Equations *763*
17.5 Calculus of Vector-Valued Functions *772*
17.6 Length of Arc *779*
17.7 Plane Motion *785*
17.8 The Unit Tangent and Unit Normal Vectors and Arc Length as a Parameter *792*
17.9 Curvature *796*
17.10 Tangential and Normal Components of Acceleration *804*

Chapter 18
VECTORS IN THREE-DIMENSIONAL SPACE AND SOLID ANALYTIC GEOMETRY
page 810

18.1 R^3, The Three-Dimensional Number Space *811*
18.2 Vectors in Three-Dimensional Space *818*
18.3 The Dot Product in V_3 *825*
18.4 Planes *829*
18.5 Lines in R^3 *836*
18.6 Cross Product *842*
18.7 Cylinders and Surfaces of Revolution *852*

18.8 Quadric Surfaces *858*
18.9 Curves in R^3 *864*
18.10 Cylindrical and Spherical Coordinates *872*

Chapter 19
DIFFERENTIAL CALCULUS OF FUNCTIONS OF SEVERAL VARIABLES
page 880

19.1 Functions of More Than One Variable *881*
19.2 Limits of Functions of More Than One Variable *889*
19.3 Continuity of Functions of More Than One Variable *900*
19.4 Partial Derivatives *905*
19.5 Differentiability and the Total Differential *913*
19.6 The Chain Rule *926*
19.7 Higher-Order Partial Derivatives *934*

Chapter 20
DIRECTIONAL DERIVATIVES, GRADIENTS, APPLICATIONS OF PARTIAL DERIVATIVES, AND LINE INTEGRALS
page 944

20.1 Directional Derivatives and Gradients *945*
20.2 Tangent Planes and Normals to Surfaces *953*
20.3 Extrema of Functions of Two Variables *956*
20.4 Some Applications of Partial Derivatives to Economics *967*
20.5 Obtaining a Function from Its Gradient *976*
20.6 Line Integrals *981*
20.7 Line Integrals Independent of the Path *989*

Chapter 21
MULTIPLE INTEGRATION
page 1001

21.1 The Double Integral *1002*
21.2 Evaluation of Double Integrals and Iterated Integrals *1008*
21.3 Center of Mass and Moments of Inertia *1016*
21.4 The Double Integral in Polar Coordinates *1022*
21.5 Area of a Surface *1028*
21.6 The Triple Integral *1034*
21.7 The Triple Integral in Cylindrical and Spherical Coordinates *1039*

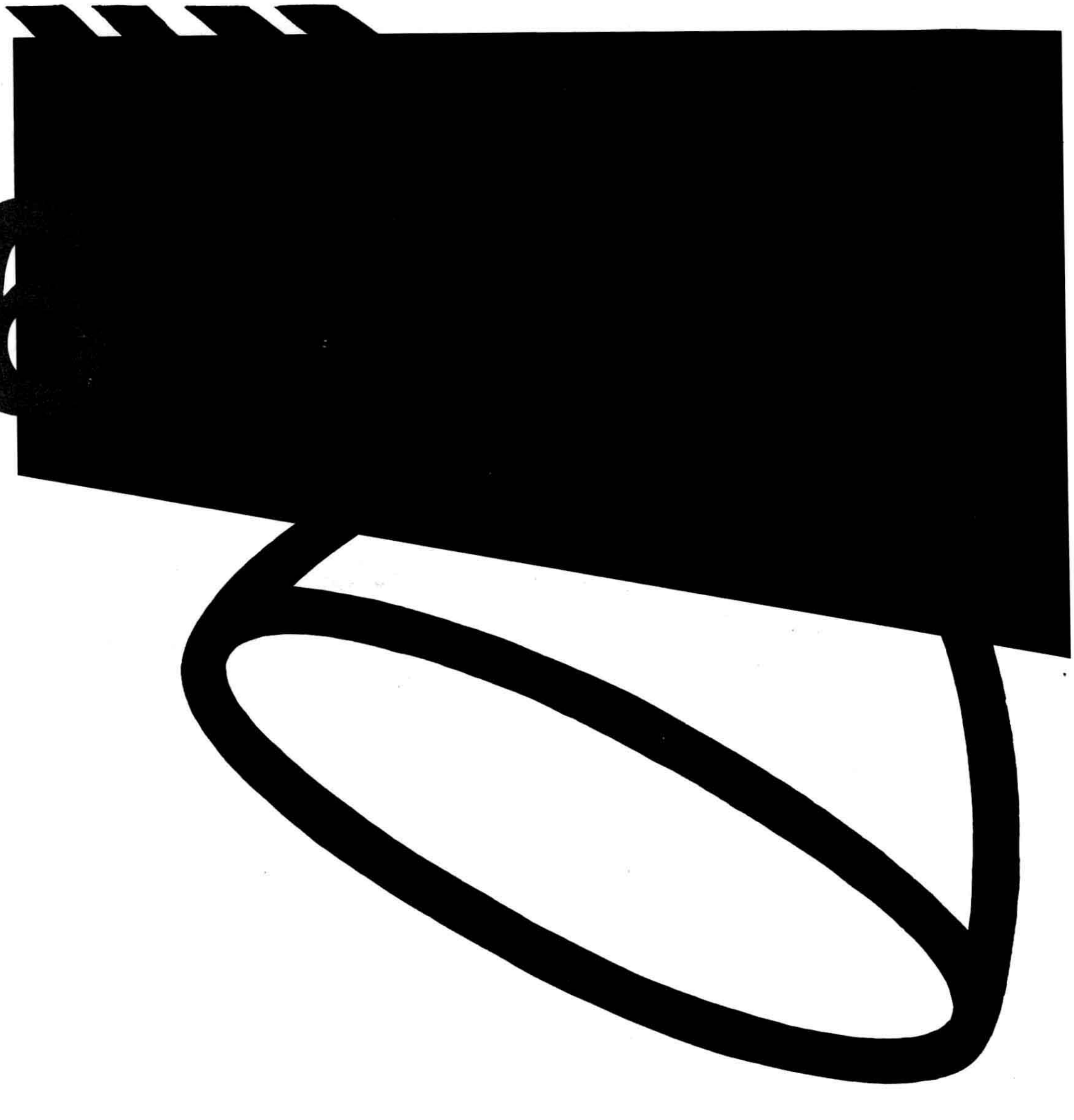
APPENDIX
page A-1

Table 1 Powers and Roots *A-2*
Table 2 Natural Logarithms *A-3*
Table 3 Exponential Functions *A-5*
Table 4 Hyperbolic Functions *A-12*
Table 5 Trigonometric Functions *A-13*
Table 6 Common Logarithms *A-14*
Table 7 The Greek Alphabet *A-16*

ANSWERS TO ODD-NUMBERED EXERCISES *A-17*

INDEX *A-25*

16



Infinite series

16.1 SEQUENCES

You have undoubtedly encountered sequences of numbers in your previous study of mathematics. For example, the numbers 5, 7, 9, 11, 13, 15 define a sequence. This sequence is said to be *finite* because there is a first and last number. If the set of numbers which defines a sequence does not have both a first and last number, the sequence is said to be *infinite*. For example, the sequence defined by

$$\tfrac{1}{3}, \tfrac{2}{5}, \tfrac{3}{7}, \tfrac{4}{9}, \ldots \tag{1}$$

is infinite because the three dots with no number following indicate that there is no last number. We are concerned here with infinite sequences, and when we use the word "sequence" it is understood that we are referring to an infinite sequence. We define a sequence as a particular kind of function.

16.1.1 Definition A *sequence* is a function whose domain is the set of positive integers.

The numbers in the range of the sequence, which are called the *elements* of the sequence, are restricted to real numbers in this book.

If the nth element is given by $f(n)$, then the sequence is the set of ordered pairs of the form $(n, f(n))$, where n is a positive integer.

• ILLUSTRATION 1: If $f(n) = n/(2n + 1)$, then

$$f(1) = \tfrac{1}{3} \qquad f(2) = \tfrac{2}{5} \qquad f(3) = \tfrac{3}{7} \qquad f(4) = \tfrac{4}{9}$$

and so on. We see that the range of f consists of the elements of sequence (1). Some of the ordered pairs in the sequence f are $(1, \tfrac{1}{3})$, $(2, \tfrac{2}{5})$, $(3, \tfrac{3}{7})$, $(4, \tfrac{4}{9})$, and $(5, \tfrac{5}{11})$. A sketch of the graph of this sequence is shown in Fig. 16.1.1. •

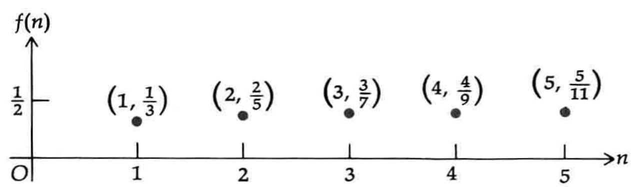


Figure 16.1.1

Usually the nth element $f(n)$ of the sequence is stated when the elements are listed in order. Thus, for the elements of sequence (1) we would write

$$\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \ldots, \frac{n}{2n+1}, \ldots$$

Because the domain of every sequence is the same, we can use the notation $\{f(n)\}$ to denote a sequence. So the sequence (1) can be denoted by $\{n/(2n+1)\}$. We also use the subscript notation $\{a_n\}$ to denote the sequence for which $f(n) = a_n$.

You should distinguish between the elements of a sequence and the sequence itself, as shown in the following illustration.

• ILLUSTRATION 2: The sequence $\{1/n\}$ has as its elements the reciprocals of the positive integers

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots \tag{2}$$

The sequence for which

$$f(n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ \dfrac{2}{n+2} & \text{if } n \text{ is even} \end{cases}$$

has as its elements

$$1, \tfrac{1}{2}, 1, \tfrac{1}{3}, 1, \tfrac{1}{4}, \ldots \tag{3}$$

The elements of sequences (2) and (3) are the same; however, the sequences are different. Sketches of the graphs of sequences (2) and (3) are shown in Figs. 16.1.2 and 16.1.3, respectively. •

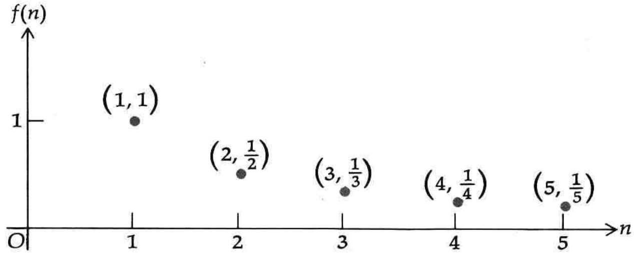


Figure 16.1.2

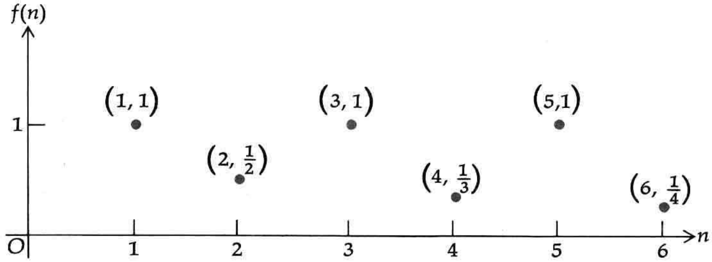


Figure 16.1.3

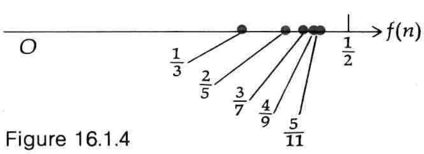


Figure 16.1.4

We now plot on a horizontal axis the points corresponding to successive elements of a sequence. This is done in Fig. 16.1.4 for sequence (1) which is $\{n/(2n+1)\}$. We see that the successive elements of the sequence get closer and closer to $\frac{1}{2}$, even though no element in the sequence has the value $\frac{1}{2}$. Intuitively we see that the element will be as close to $\frac{1}{2}$ as we please by taking the number of the element sufficiently large. Or stating this another way, we can make $|n/(2n+1) - \frac{1}{2}|$ less than any given ϵ by taking n large enough. Because of this we state that the limit of the sequence $\{n/(2n+1)\}$ is $\frac{1}{2}$.

In general, if there is a number L such that $|a_n - L|$ is arbitrarily small for n sufficiently large, we say the sequence $\{a_n\}$ has the limit L. Following is the precise definition of the limit of a sequence.

16.1.2 Definition A sequence $\{a_n\}$ is said to have the limit L if for every $\epsilon > 0$ there exists a number $N > 0$ such that $|a_n - L| < \epsilon$ for every integer $n > N$; and we write

$$\lim_{n \to +\infty} a_n = L$$

EXAMPLE 1: Use Definition 16.1.2 to prove that the sequence

$$\left\{\frac{n}{2n+1}\right\}$$

has the limit $\frac{1}{2}$.

SOLUTION: We must show that for any $\epsilon > 0$ there exists a number $N > 0$ such that

$$\left|\frac{n}{2n+1} - \frac{1}{2}\right| < \epsilon \qquad \text{for every integer } n > N$$

$$\left|\frac{n}{2n+1} - \frac{1}{2}\right| = \left|\frac{2n-2n-1}{2(2n+1)}\right| = \left|\frac{-1}{4n+2}\right| = \frac{1}{4n+2}$$

Hence, we must find a number $N > 0$ such that

$$\frac{1}{4n+2} < \epsilon \qquad \text{for every integer } n > N$$

But

$$\frac{1}{4n+2} < \epsilon \quad \text{is equivalent to} \quad 2n+1 > \frac{1}{2\epsilon}$$

which is equivalent to

$$n > \frac{1-2\epsilon}{4\epsilon}$$

So it follows that

$$\left|\frac{n}{2n+1} - \frac{1}{2}\right| < \epsilon \qquad \text{for every integer } n > \frac{1-2\epsilon}{4\epsilon}$$

Therefore, if $N = (1-2\epsilon)/4\epsilon$, Definition 16.1.2 holds.

In particular, if $\epsilon = \frac{1}{8}$, $N = (1-\frac{1}{4})/\frac{1}{2} = \frac{3}{2}$. So

$$\left|\frac{n}{2n+1} - \frac{1}{2}\right| < \frac{1}{8} \qquad \text{for every integer } n > \frac{3}{2}$$

For instance, if $n = 4$,

$$\left|\frac{n}{2n+1} - \frac{1}{2}\right| = \left|\frac{4}{9} - \frac{1}{2}\right| = \left|\frac{-1}{18}\right| = \frac{1}{18} < \frac{1}{8}$$

• ILLUSTRATION 3: Consider the sequence $\{(-1)^{n+1}/n\}$. Note that the nth element of this sequence is $(-1)^{n+1}/n$ and $(-1)^{n+1}$ is equal to $+1$ when n is odd and to -1 when n is even. Hence, the elements of the sequence can be written

$$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots, \frac{(-1)^{n+1}}{n}, \ldots$$

In Fig. 16.1.5 are plotted points corresponding to successive elements of this sequence. In the figure, $a_1 = 1$, $a_2 = -\frac{1}{2}$, $a_3 = \frac{1}{3}$, $a_4 = -\frac{1}{4}$, $a_5 = \frac{1}{5}$, $a_6 = -\frac{1}{6}$, $a_7 = \frac{1}{7}$, $a_8 = -\frac{1}{8}$, $a_9 = \frac{1}{9}$, $a_{10} = -\frac{1}{10}$. The limit of the sequence is 0 and the elements oscillate about 0. •

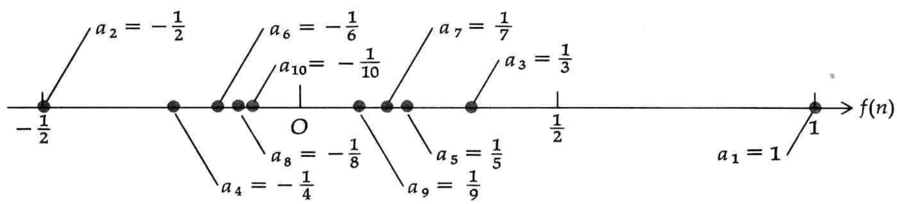


Figure 16.1.5

Compare Definition 16.1.2 with Definition 4.1.1 of the limit of $f(x)$ as x increases without bound. The two definitions are almost identical; however, when we state that $\lim_{x\to+\infty} f(x) = L$, the function f is defined for all real numbers greater than some real number R, while when we consider $\lim_{n\to+\infty} a_n$, n is restricted to positive integers. We have, however, the following theorem which follows immediately from Definition 4.1.1.

16.1.3 Theorem If $\lim_{x\to+\infty} f(x) = L$, and f is defined for every positive integer, then also $\lim_{n\to+\infty} f(n) = L$ when n is any positive integer.

The proof is left as an exercise (see Exercise 20).

• ILLUSTRATION 4: We verify Theorem 16.1.3 for the sequence of Example 1. For that sequence $f(n) = n/(2n+1)$. Hence, $f(x) = x/(2x+1)$ and

$$\lim_{x\to+\infty} \frac{x}{2x+1} = \lim_{x\to+\infty} \frac{1}{2+\frac{1}{x}} = \frac{1}{2}$$

It follows then from Theorem 16.1.3 that $\lim_{n\to+\infty} f(n) = \frac{1}{2}$ when n is any positive integer. This agrees with the solution of Example 1. •

16.1.4 Definition If a sequence $\{a_n\}$ has a limit, the sequence is said to be *convergent*, and we say that a_n *converges* to that limit. If the sequence is not convergent, it is said to be *divergent*.

EXAMPLE 2: Determine if the sequence

$$\left\{\frac{4n^2}{2n^2+1}\right\}$$

is convergent or divergent.

SOLUTION: We wish to determine if $\lim_{n\to+\infty} 4n^2/(2n^2+1)$ exists. Let $f(x) = 4x^2/(2x^2+1)$ and investigate $\lim_{x\to+\infty} f(x)$.

$$\lim_{x\to+\infty} \frac{4x^2}{2x^2+1} = \lim_{x\to+\infty} \frac{4}{2+\dfrac{1}{x^2}} = 2$$

Therefore, by Theorem 16.1.3, $\lim_{n\to+\infty} f(n) = 2$. We conclude that the given sequence is convergent and that $4n^2/(2n^2+1)$ converges to 2.

EXAMPLE 3: Prove that if $|r| < 1$, the sequence $\{r^n\}$ is convergent and that r^n converges to zero.

SOLUTION: First of all, if $r = 0$, the sequence is $\{0\}$ and $\lim_{x\to+\infty} 0 = 0$. Hence, the sequence is convergent and the nth element converges to zero.

If $0 < |r| < 1$, we consider the function f defined by $f(x) = r^x$, where x is any positive number, and show that $\lim_{x\to+\infty} r^x = 0$. Then from Theorem 16.1.3 it will follow that $\lim_{n\to+\infty} r^n = 0$ when n is any positive integer.

To prove that $\lim_{x\to+\infty} r^x = 0$ $(0 < |r| < 1)$, we shall show that for any $\epsilon > 0$ there exists a number $N > 0$ such that

$$|r^x - 0| < \epsilon \qquad \text{whenever } x > N \tag{4}$$

Statement (4) is equivalent to

$$|r|^x < \epsilon \qquad \text{whenever } x > N$$

which is true if and only if

$$\ln |r|^x < \ln \epsilon \qquad \text{whenever } x > N$$

or, equivalently,

$$x \ln |r| < \ln \epsilon \qquad \text{whenever } x > N \tag{5}$$

Because $0 < |r| < 1$, $\ln |r| < 0$. Thus, (5) is equivalent to

$$x > \frac{\ln \epsilon}{\ln |r|} \qquad \text{whenever } x > N$$

Therefore, if we take $N = \ln \epsilon / \ln |r|$, we may conclude (4). Consequently, $\lim_{x\to+\infty} r^x = 0$, and so $\lim_{n\to+\infty} r^n = 0$ if n is any positive integer. Hence, by Definitions 16.1.2 and 16.1.4, $\{r^n\}$ is convergent and r^n converges to zero.

EXAMPLE 4: Determine if the sequence $\{(-1)^n + 1\}$ is convergent or divergent.

SOLUTION: The elements of this sequence are 0, 2, 0, 2, 0, 2, . . . , $(-1)^n + 1$, ... Because $a_n = 0$ if n is odd, and $a_n = 2$ if n is even, it appears that the sequence is divergent. To prove this, let us assume that the sequence is convergent and show that this assumption leads to a contradiction. If the sequence has the limit L, then by Definition 16.1.2, for every $\epsilon > 0$ there exists a number $N > 0$ such that $|a_n - L| < \epsilon$ for every integer $n > N$. In particular, when $\epsilon = \frac{1}{2}$, there exists a number $N > 0$ such that

$$|a_n - L| < \tfrac{1}{2} \qquad \text{for every integer } n > N$$

or, equivalently,

$$-\tfrac{1}{2} < a_n - L < \tfrac{1}{2} \qquad \text{for every integer } n > N \tag{6}$$

Because $a_n = 0$ if n is odd and $a_n = 2$ if n is even, it follows from (6) that

$$-\tfrac{1}{2} < -L < \tfrac{1}{2} \quad \text{and} \quad -\tfrac{1}{2} < 2 - L < \tfrac{1}{2}$$

But if $-L > -\frac{1}{2}$, then $2 - L > \frac{3}{2}$; hence, $2 - L$ cannot be less than $\frac{1}{2}$. So we have a contradiction, and therefore the given sequence is divergent.

EXAMPLE 5: Determine if the sequence

$$\left\{n \sin \frac{\pi}{n}\right\}$$

is convergent or divergent.

SOLUTION: We wish to determine if $\lim_{n\to+\infty} n \sin(\pi/n)$ exists. Let $f(x) = x \sin(\pi/x)$ and investigate $\lim_{x\to+\infty} f(x)$. Because $f(x)$ can be written as $[\sin(\pi/x)]/(1/x)$ and $\lim_{x\to+\infty} \sin(\pi/x) = 0$ and $\lim_{x\to+\infty} (1/x) = 0$, we can apply L'Hôpital's rule and obtain

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{-\dfrac{\pi}{x^2}\cos\dfrac{\pi}{x}}{-\dfrac{1}{x^2}} = \lim_{x\to+\infty} \pi \cos\frac{\pi}{x} = \pi$$

Therefore, $\lim_{n\to+\infty} f(n) = \pi$ when n is a positive integer. So the given sequence is convergent and $n \sin(\pi/n)$ converges to π.

We have limit theorems for sequences, which are analogous to limit theorems for functions given in Chapter 2. We state these theorems by using the terminology of sequences. The proofs are omitted because they are almost identical to the proofs of the corresponding theorems given in Chapter 2.

16.1.5 Theorem If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

(i) the constant sequence $\{c\}$ has c as its limit;
(ii) $\lim_{n\to+\infty} ca_n = c \lim_{n\to+\infty} a_n$;
(iii) $\lim_{n\to+\infty} (a_n \pm b_n) = \lim_{n\to+\infty} a_n \pm \lim_{n\to+\infty} b_n$;