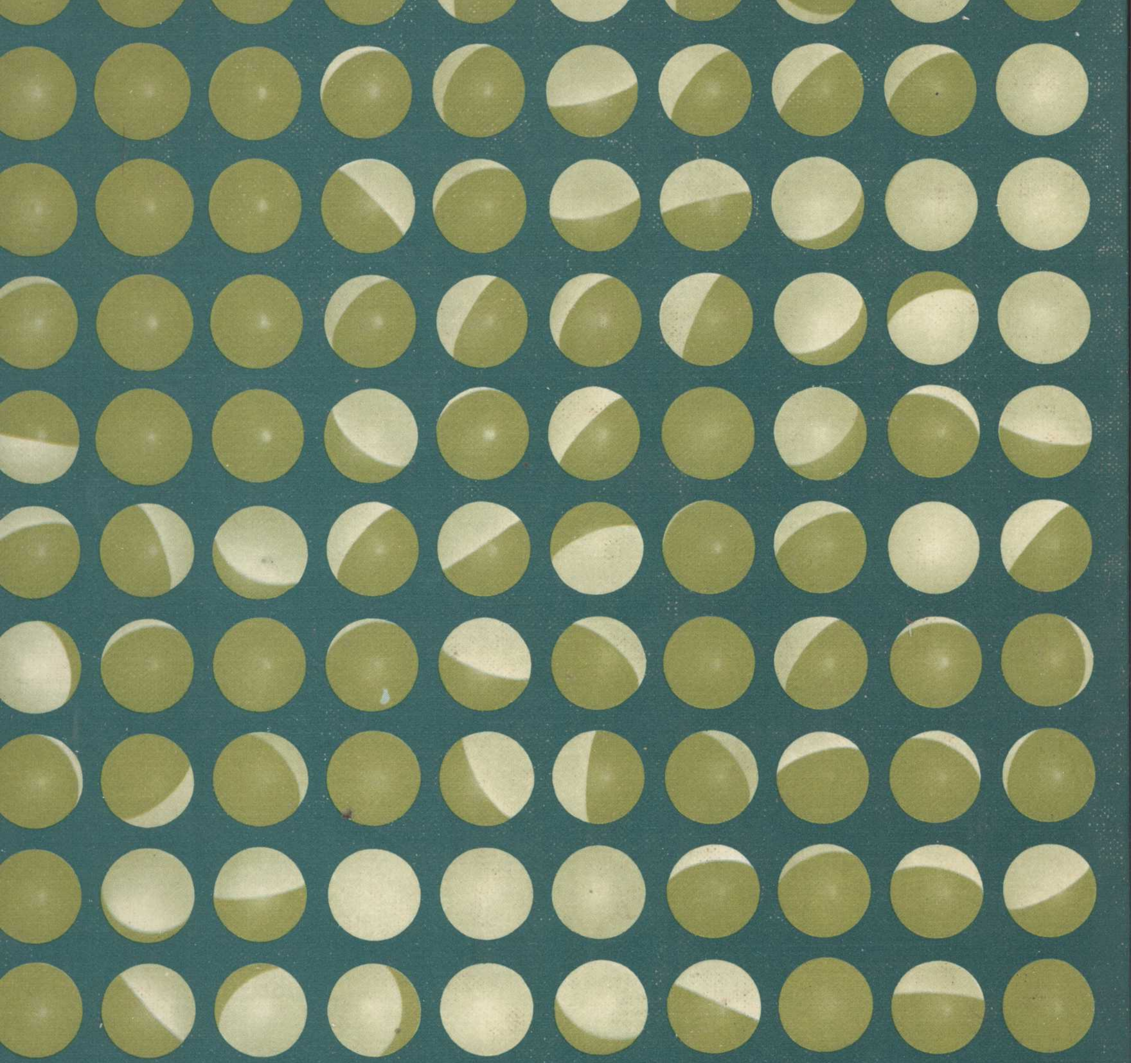

algebra

FRANK J. FLEMING

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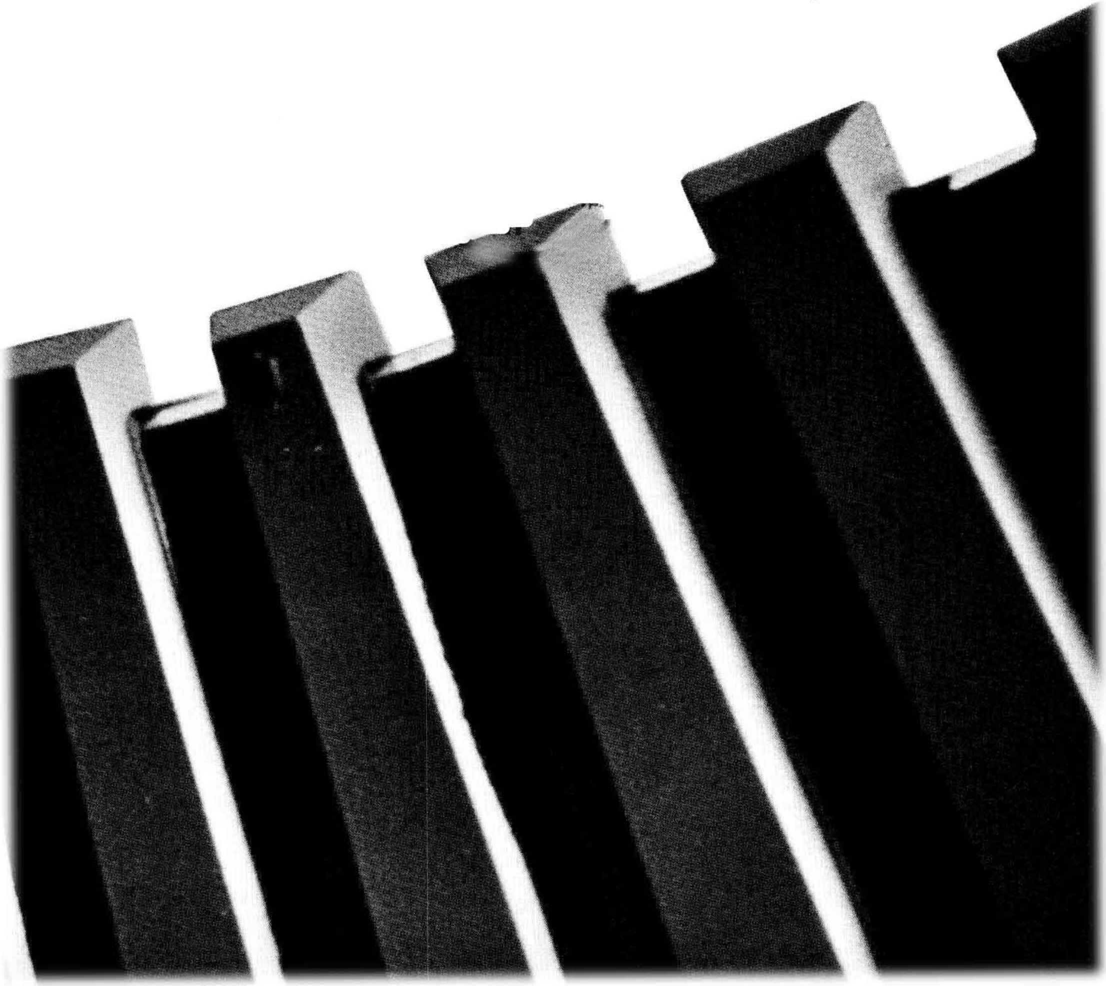

COVER: Kugelbild K 100b. A kinetic sculpture by Paul Talman. Courtesy of the artist.

TITLE PAGE: Detail of staircase, Banco Wiese, Lima, Peru. Architect: Fernando Osma. Photograph by Mauro Mujica.

Acknowledgments for other photographs on page 483.

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Preface

Traditionally, "intermediate" algebra courses concentrate on strengthening manipulative skills introduced in elementary algebra courses. While this emphasis is desirable, experience shows that many students completing elementary algebra courses also need a better understanding of the basic principles of algebra if they are to succeed in more advanced mathematics courses, particularly in calculus. To solve both problems, this book maintains a balance between manipulative drills and abstract theory.

A seemingly large amount of review material in a course often makes self-motivation difficult. Students tend to mistake familiarity for understanding and may not concentrate on their studies. This book challenges students with new approaches to old material.

The axiomatic structure of algebra is developed by combining both rigorous and intuitive reasoning to help foster a better understanding of the structure of algebra. Several topics that are usually introduced in an advanced algebra course are woven into the earlier chapters to encourage higher achievement. Other topics are given partial or full chapter treatment, as will be noted later. Hence, this book will serve either for a course in intermediate algebra or a course in college algebra, depending on the goals of the teacher.

The exercise sets that appear at the end of each section have been carefully constructed so that every idea is adequately covered. The problems are arranged with increasing difficulty from the beginning to the end of each set.

Numerous worked-out examples have been included, both in the text and in the exercise sets, to illustrate the discussion.

Chapter 1 reviews the concept of sets and operations on sets, with particular emphasis on some properties of sets with respect to the operations of union and intersection. This discussion leads to a review of the properties of real numbers with respect to the operations of addition and multiplication. The method of deductive proof is re-emphasized by carefully leading the student through the proof of selected theorems.

Chapters 2 and 4 are primarily a review of similar topics in an elementary course, with particular attention paid to equivalent equations and inequalities. Quadratic inequalities are introduced. Their solutions are determined by analysis rather than by mechanical means. Chapter 3 is largely manipulative. However, more attention is given to reasoning than to simple computational procedures.

Binary relations are introduced in Chapter 5 in order to develop the concept of a function. The function concept is then used throughout the remainder of the book wherever applicable. Graphs of relations and functions are also discussed in Chapter 5, with the emphasis on developing techniques of quick sketching rather than tedious point plotting. These techniques are then applied in Chapter 6, where the properties of the conic sections and graphs of inequalities are discussed. Chapter 6 also includes discussions of variation and inverse functions.

The material in Chapters 7 and 8, polynomials and rational expressions, which are treated earlier in most textbooks, is included at this point so that the discussion of polynomial functions and rational functions can be made in a more logical sequence.

Chapter 9, Systems of Equations and Inequalities, goes beyond the usual treatment to emphasize the generation of equivalent systems, both by the substitution technique and by analytic methods. Systems of inequalities are solved by graphic methods.

The remaining four chapters deal with topics usually covered in an intermediate course without prior introduction in elementary algebra. In Chapter 10, the properties of complex numbers are developed through the algebra of ordered pairs of real numbers. This approach leads to an introduction of vectors and vector spaces. Chapter 11 contains sufficient elementary matrix algebra to introduce the determinant function and to discuss the properties of determinants.

Logarithmic functions and computations with logarithms are developed in Chapter 12 from exponential functions and the properties of exponents. This discussion emphasizes the independence of the characteristic and the mantissa of the common logarithm of a number.

The sequence function and sequences are discussed in Chapter 13. Arithmetic and geometric sequences are treated in detail. The chapter concludes with an intuitive approach to the binomial expansion.

I am particularly grateful to Dr. James L. Murphy of Michigan State University and to Professor Monroe Kline and other members of the mathematics department of Los Angeles Pierce College for their helpful criticism—and also to my wife Marilyn, who typed the manuscript.

Frank J. Fleming

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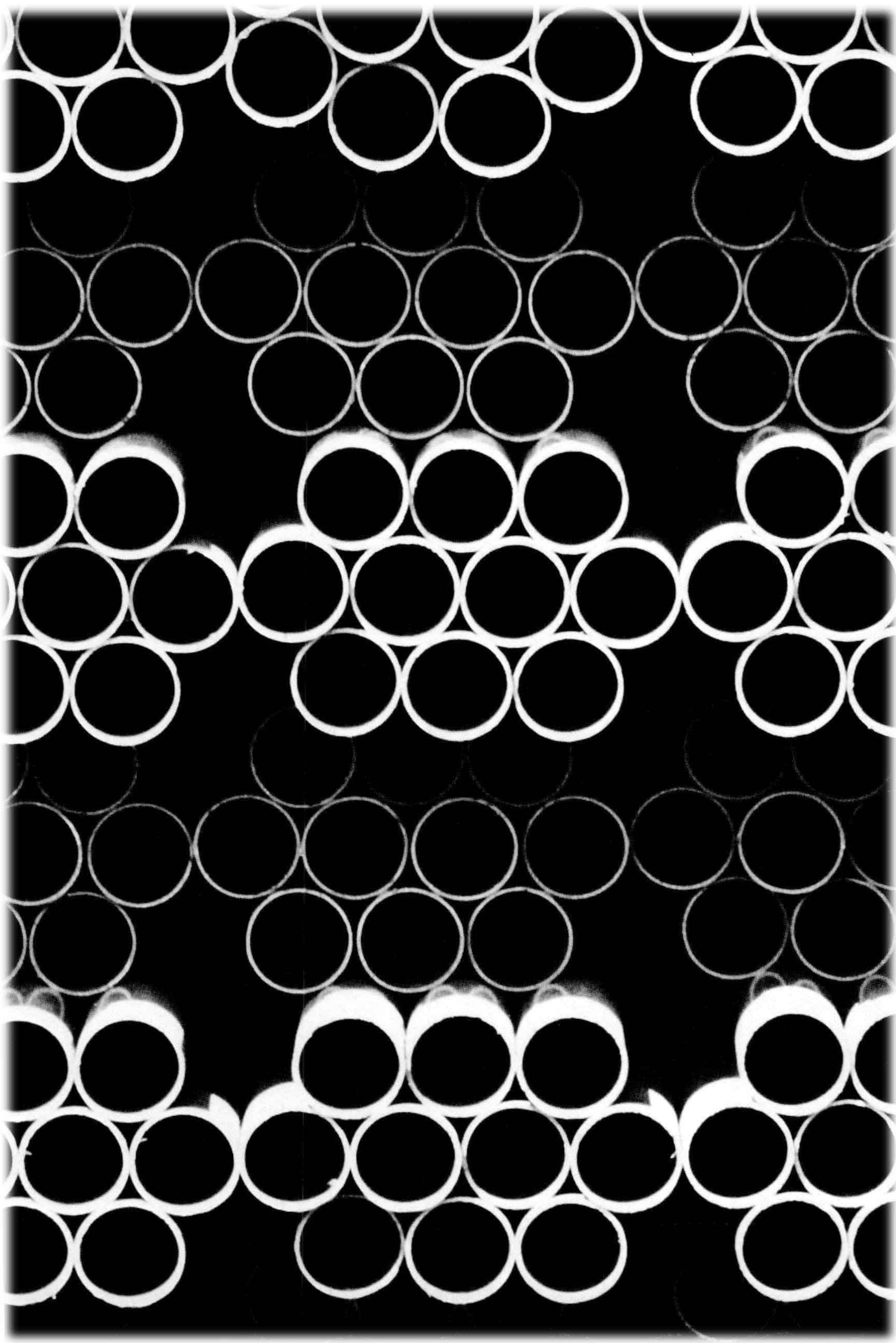
algebra



1 / The System of Real Numbers

In a modern elementary algebra course, the system of real numbers is developed by introducing the system of natural numbers and then extending it to include other kinds of numbers, as the need arises. It is not our intent to repeat this development, but there are some basic concepts that, we feel, should be reemphasized. We shall characterize the real numbers as being in one-to-one correspondence with the points on the number line; that is, every real number can be associated with one and only one point on the line, and every point on the line can be associated with one and only one real number.

1.1 Sets

The term **set** is used to refer to any well-defined collection of discrete "objects." The "objects," or **elements** of the set, may be tangible things such as books, people, or symbols, or intangibles such as ideas. A set is well defined if it is possible to determine whether or not a particular object is a member of the given set. Sets may be *identified,* or *specified,* by listing their elements, by clearly describing the conditions for set membership, or by using "set-builder" notation.

Examples Describe the elements of the following sets.

(a) The set composed of the first five letters of the English alphabet.
(b) The set of pine trees in California.
(c) The set of integers between 1 and 10.

Solutions

(a) $\{a, b, c, d, e\}$ is a listing or roster of the elements.
(b) {pine trees in California} is a description or rule of set membership.
(c) $\{x|1 < x < 10,\ x \in \boldsymbol{J}\}$ is set-builder notation, which is read: "The set of all x such that 1 is less than x, which is less than 10, and x is an integer."

NOTE: We assume that you are already familiar with the various symbols used in set notation. For review purposes, a list of symbols and their meanings is given inside the front cover.

Any symbol used to represent an unspecified element in a set containing two or more members is called a **variable**. The set whose elements are so represented is called the **replacement set** for the variable. Thus, in Example (c) above, x is a variable, and $\{2, 3, 4, 5, 6, 7, 8, 9\}$ is the replacement set for x. A symbol used to represent the element in a set that has exactly one member is called a **constant**.

Here we have assumed that it is sometimes possible to count the elements in a given set. Numbers used for counting are called **natural numbers** and the set of such numbers is designated by $\boldsymbol{N}$. The number that specifies how many elements there are in a set is called the **cardinality** of the set and such a number is said to be used in a **cardinal** sense.

The set of numbers representing the cardinalities of all sets whose elements can be counted is the set of **whole numbers** $\boldsymbol{W}$. The set that contains no elements (cardinality zero) is the **empty set**, or the **null set**, and is represented by the symbol $\varnothing$.

The sets of numbers with which you should be familiar are:

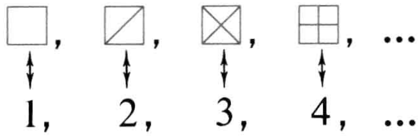
$$\begin{aligned} \boldsymbol{N} &= \{\text{natural numbers}\} \\ \boldsymbol{W} &= \{\text{whole numbers}\} \\ \boldsymbol{J} &= \{\text{integers}\} \\ \boldsymbol{Q} &= \{\text{rational numbers}\} \\ \boldsymbol{H} &= \{\text{irrational numbers}\} \\ \boldsymbol{R} &= \{\text{real numbers}\} \end{aligned}$$

In any study of algebra we are concerned with the relationship that may or may not exist between two sets or between the elements of one or more sets. These relationships involve such things as equality, equivalence, order, etc.

Two sets are **equal**, or identical, if and only if they contain the same elements. Two sets are **equivalent** if they have the same cardinality, that is, if a one-to-one correspondence can be shown to exist between the members of the two sets. Thus, equal sets are equivalent, but equivalent sets are not necessarily equal.

In general, the order in which the elements of a set are listed is not important. If some particular order is specified, then the set is an **ordered set**. The

Figure 1.1-1



natural numbers and the whole numbers are ordered sets. One way in which the elements of some sets can be ordered is by placing them in one-to-one correspondence with N as suggested in Figure 1.1-1. However, this does not hold for all sets. If there is a last element when the elements of a set are placed in correspondence with the natural numbers, the set is said to be **finite**. If there is no such last element, then the set is called **infinite**. In listing the elements of an infinite set, only a few elements are listed. These elements are followed, or preceded, by three dots that represent the words "and so on."

Examples

(a) $N = \{1, 2, 3, \ldots\}$
(b) {Negative integers} $= \{\ldots, -3, -2, -1\}$
(c) $J = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

An important relationship that may exist between two sets is the **subset relation**. Of two sets A and B, set A is said to be a **subset** of the set B, or $A \subseteq B$, provided every element of A is also a member of B. Thus, if $A = B$, $A \subseteq B$. Also for every set A, $A \subseteq A$; that is, every set is a subset of itself. If B contains at least one element that is not a member of A, then A is called a **proper subset** of B or $A \subset B$. The symbol $\not\subseteq$ indicates that one set is not a subset of a second set.

Examples

Place the proper symbol $\subseteq$, $\subset$, or $\not\subseteq$ between the two sets in each of the following pairs.

(a) $\{1, 2, 3, 4\}$ $\qquad \{1, 2, 3, 4, 5\}$
(b) $\{a, b, c, d, e\}$ $\qquad \{a, b, c, d\}$

Solutions

(a) $\{1, 2, 3, 4\} \subset \{1, 2, 3, 4, 5\}$ or $\{1, 2, 3, 4\} \subseteq \{1, 2, 3, 4, 5\}$
(b) $\{a, b, c, d, e\} \not\subseteq \{a, b, c, d\}$

Example

What is the subset of A that contains all positive integers in A if

$A = \{-5, -\frac{2}{3}, 0, 1, \frac{6}{3}, \sqrt{25}, \frac{17}{2}, \sqrt{30}\}$?

Solution

$\{1, \frac{6}{3}, \sqrt{25}\}$

In general, we shall have little occasion to distinguish between a *subset* and a *proper subset,* and we shall ordinarily use the symbol $\subseteq$. As an example, we can apply the subset relation to the sets of real numbers, so that

$$\boldsymbol{N} \subseteq \boldsymbol{W} \subseteq \boldsymbol{J} \subseteq \boldsymbol{Q} \subseteq \boldsymbol{R} \quad \text{and} \quad \boldsymbol{H} \subseteq \boldsymbol{R}.$$

EXERCISE 1.1

1. Define the following terms.
(a) variable
(b) replacement set
(c) constant
(d) equal sets
(e) subset

2. Define the following terms.
(a) cardinal use of a number
(b) infinite set
(c) finite set
(d) proper subset

In Problems 3–12 place the proper symbol $\in$ or $\notin$ in the space between each pair of elements and sets.

Examples (a) x {letters of the alphabet} (b) 8 $\{1, 3, 5, \ldots\}$

Solutions (a) $x \in$ {letters of the alphabet} (b) $8 \notin \{1, 3, 5, \ldots\}$

3. a $\{a, b, c\}$
4. n $\{a, e, i, o, u\}$
5. $\frac{2}{3}$ {natural numbers}
6. 1492 {whole numbers}
7. 3 $\{x|3 < x < 5, x \in \boldsymbol{W}\}$
8. 4 $\{y|5 > y > 2, y \in \boldsymbol{N}\}$
9. Monday {days of the week}
10. August {months with thirty days}
11. ⍁ {⦶, ⊖, ⊕, ⊠}
12. ⋇ {⋇, +, □, ×}

Let $A = \{1, 2, 3\}$, $B = \{1, 3, 2\}$, $C = \{1, 2\}$, $D = \{1, 3\}$, $E = \{2\}$. In Problems 13–22, replace the comma in each pair with the correct symbol $\subseteq$ or $\nsubseteq$.

13. A, B
14. C, A
15. E, D
16. $2, A$
17. C, D
18. B, C
19. $3, A$
20. A, E
21. $\{2, 3\}, A$
22. $\{2\}, A$

Justify your answers in Problems 23–30.

23. If $A \subseteq B$ and $x \in A$, must x be an element of B?
24. If $A \subseteq B$ and $y \in B$, must y be an element of A?
25. If $P \subseteq Q$ and $Q \subseteq P$, in what other way is P related to Q?
26. If $P \subseteq Q$ and $Q \subseteq R$, in what way is P related to R?
27. If $A \subseteq B$ and $C \subseteq B$, must A be a subset of C?
28. If $A \subseteq B$ and $C \subseteq B$, can A be a subset of C?
29. If $P = Q$ and $Q = R$, in what way are P and R always related?
30. If $P = Q$ and $Q \subseteq R$, in what way are P and R always related?