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Nonlinear Differential Equations of Monotone Types in Banach Spaces

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Preface

In the last decades, functional methods played an increasing role in the qualitative theory of partial differential equations. The spectral methods and theory of C_0-semigroups of linear operators as well as Leray–Schauder degree theory, fixed point theorems, and theory of maximal monotone nonlinear operators are now essential functional tools for the treatment of linear and nonlinear boundary value problems associated with partial differential equations.

An important step was the extension in the early seventies of the nonlinear dynamics of accretive (dissipative) type of the Hille–Yosida theory of C_0-semigroups of linear continuous operators. The main achievement was that the Cauchy problem associated with nonlinear m-accretive operators in Banach spaces is well posed and the corresponding dynamic is expressed by the Peano exponential formula from finite-dimensional theory. This fundamental result is the corner stone of the whole existence theory of nonlinear infinite dynamics of dissipative type and its contribution to the development of the modern theory of nonlinear partial differential equations cannot be underestimated.

Previously, in mid-sixties, some spectacular properties of maximal monotone operators and their close relationship with convex analysis and m-accretivity were revealed. In fact, Minty's discovery that in Hilbert spaces nonlinear maximal monotone operators coincide with m-accretive operators was crucial for the development of the theory. Although with respect to the middle and end of the seventies, little new material on this subject has come to light, the field of applications grew up and still remains in actuality. In the meantime, some excellent monographs were published where these topics were treated exhaustively and with extensive bibliographical references. Here, we confine ourselves to the presentation of basic results of the theory of nonlinear operators of monotone type and the corresponding dynamics generated in Banach spaces. These subjects were also treated in the author's books *Nonlinear Semigroups and Differential Equations in Banach Spaces* (Noordhoff, 1976) and *Analysis and Control of Nonlinear Infinite Dimensional Systems* (Academic Press, 1993), but the present book is more oriented to applications. We refrain from an exhaustive treatment or details that would divert us from the principal aim of this book: the presentation of essential results of the theory and its illustration by significant problems of nonlinear partial differential equations. Our aim is to present functional tools for the study of a large class of nonlinear problems and open to the reader the way towards applications. This book can serve as a teaching text for graduate students and it is self-contained. One assumes, however, basic knowledge of real and functional analysis as well as of differential equations. The literature on this argument is so vast and accessible in print that I have dispensed with detailed references or bibliographical comments. I have confined myself to a selected bibliography arranged at the end of each chapter.

The present book is based on a graduate course given by the author at the University of Iaşi during the past twenty years as well as on one-semester graduate courses at the University of Virginia in 2005 and the University of Trento in 2008.

In the preparation of the present book, I have received valuable help from my colleagues, Ioan Vrabie and Cătălin Lefter (Al.I. Cuza University of Iaşi), Gabriela Marinoschi (Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy) and Luca Lorenzi from University of Parma, who read the preliminary draft of the book and made numerous comments and suggestions which have permitted me to improve the presentation and correct the errors. Elena Mocanu from the Institute of Mathematics in Iaşi has done a great job in preparing and processing this text for printing.

Iaşi, September 2009 *Viorel Barbu*

Acronyms

$\mathbf{R}$	the real line $(-\infty,\infty)$
$\mathbf{R}^N$	the N-dimensional Euclidean space
$\mathbf{R}^+ = (0,+\infty)$,	
$\mathbf{R}^- = (-\infty,0)$,	
$\overline{\mathbf{R}} = (-\infty,+\infty]$,	
$\mathbf{R}^N_+ = \{(x_1,\ldots,x_N); x_N > 0\}$	
Ω	open subset of $\mathbf{R}^N$
$\partial\Omega$	the boundary of Ω
$Q = \Omega \times (0,T)$,	
$\Sigma = \partial\Omega \times (0,T)$,	where $0 < T < \infty$
$\|\cdot\|_X$	the norm of a linear normed space X
X^*	the dual of space X
$L(X,Y)$	the space of linear continuous operators from X to Y
∇f	the gradient of the map $f : X \to \mathbf{R}$
∂f	the subdifferential of $f : X \to \mathbf{R}$
B^*	the adjoint of the operator B
$\overline{C}$	the closure of the set C
$\mathrm{int}\, C$	the interior of C
$\mathrm{conv}\, C$	the convex hull of C
sign	the signum function on X : $\mathrm{sign}\, x = x/\|x\|_X$ if $x \neq 0$ $\mathrm{sign}\, 0 = \{x;\ \|x\| \leq 1\}$
$C^k(\Omega)$	the space of real-valued functions on Ω that are continuously differentiable up to order k, $0 \leq k \leq \infty$
$C^k_0(\Omega)$	the subspace of functions in $C^k(\Omega)$ with compact support in Ω
$\mathscr{D}(\Omega)$	the space $C^\infty_0(\Omega)$
$\frac{d^k u}{dt^k}$, $u^{(k)}$	the derivative of order k of $u : [a,b] \to X$
$\mathscr{D}'(\Omega)$	the dual of $\mathscr{D}(\Omega)$ (i.e., the space of distributions on Ω)
$C(\overline{\Omega})$	the space of continuous functions on $\overline{\Omega}$

$L^p(\Omega)$	the space of p-summable functions $u : \Omega \to \mathbf{R}$ endowed with the norm $\|u\|_p = (\int_\Omega |u(x)|^p dx)^{1/p}$, $1 \le p < \infty$, $\|u\|_\infty = \operatorname{ess\,sup}_{x \in \Omega} |u(x)|$ for $p = \infty$
$L^p_m(\Omega)$	the space of p-summable functions $u : \Omega \to \mathbf{R}^m$
$W^{m,p}(\Omega)$	the Sobolev space $\{u \in L^p(\Omega);\ D^\alpha u \in L^p(\Omega), \|\alpha\| \le m,\ 1 \le p \le \infty\}$
$W_0^{m,p}(\Omega)$	the closure of $C_0^\infty(\Omega)$ in the norm of $W^{m,p}(\Omega)$
$W^{-m,q}(\Omega)$	the dual of $W_0^{m,p}(\Omega)$; $(1/p) + (1/q) = 1$, $p < \infty$, $q > 1$
$H^k(\Omega), H_0^k(\Omega)$	the spaces $W^{k,2}(\Omega)$ and $W_0^{k,2}(\Omega)$, respectively
$L^p(a,b;X)$	the space of p-summable functions from (a,b) to X (Banach space) $1 \le p \le \infty$, $-\infty \le a < b \le \infty$
$AC([a,b];X)$	the space of absolutely continuous functions from $[a,b]$ to X
$BV([a,b];X)$	the space of functions with bounded variation on $[a,b]$
$BV(\Omega)$	the space of functions with bounded variation on Ω
$W^{1,p}([a,b];X)$	the space $\{u \in AC([a,b];X);\ du/dt \in L^p([a,b];X)\}$

Contents

Chapter 1
Fundamental Functional Analysis

Abstract The aim of this chapter is to provide some standard basic results pertaining to geometric properties of normed spaces, convex functions, Sobolev spaces, and variational theory of linear elliptic boundary value problems. Most of these results, which can be easily found in textbooks or monographs, are given without proof or with a sketch of proof only.

1.1 Geometry of Banach Spaces

Throughout this section X is a real normed space and X^* denotes its dual. The value of a functional $x^* \in X^*$ at $x \in X$ is denoted by either (x^*,x) or $x^*(x)$, as is convenient. The norm of X is denoted by $\|\cdot\|$, and the norm of X^* is denoted by $\|\cdot\|_*$. If there is no danger of confusion we omit the asterisk from the notation $\|\cdot\|_*$ and denote both the norms of X and X^* by the symbol $\|\cdot\|$.

We use the symbol lim or $\rightarrow$ to indicate *strong convergence* in X and w-lim or $\rightharpoonup$ for *weak convergence* in X. By w^*-lim or $\rightharpoonup$ we indicate *weak-star* convergence in X^*. The space X^* endowed with the weak-star topology is denoted by X^*_w.

Define on X the mapping $J : X \rightarrow 2^{X^*}$:

$$J(x) = \{x^* \in X^*;\ (x^*,x) = \|x\|^2 = \|x^*\|^2\}, \qquad \forall x \in X. \tag{1.1}$$

By the Hahn–Banach theorem we know that for every $x_0 \in X$ there is some $x_0^* \in X^*$ such that $(x_0^*,x_0) = \|x_0\|$ and $\|x_0^*\| \leq 1$.

Indeed, the linear functional $f : Y \rightarrow \mathbf{R}$ defined by $f(x) = \alpha\|x_0\|$ for $x = \alpha x_0$, where $Y = \{\alpha x_0;\ \alpha \in \mathbf{R}\}$, has a linear continuous extension $x_0^* \in X^*$ on X such that $|(x_0^*,x)| \leq \|x\|\ \forall x \in X$. Hence, $(x_0^*,x_0) = \|x_0\|$ and $\|x_0^*\| \leq 1$ (in fact, $\|x_0^*\| = 1$). Clearly, $x_0^*\|x_0\| \in J(x_0)$ and so $J(x_0) \neq \emptyset$ for every $x_0 \in X$.

The mapping $J : X \rightarrow X^*$ is called the *duality mapping* of the space X. In general, the duality mapping J is multivalued.

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The inverse mapping $J^{-1} : X^* \to X$ defined by $J^{-1}(x^*) = \{x \in X;\ x^* \in J(x)\}$ also satisfies

$$J^{-1}(x^*) = \{x \in X;\ \|x\| = \|x^*\|,\ (x^*,x) = \|x\|^2 = \|x^*\|^2\}.$$

If the space X is reflexive (i.e., $X = X^{**}$), then clearly J^{-1} is just the duality mapping of X^* and so $D(J^{-1}) = X^*$. As a matter of fact, reflexivity plays an important role everywhere in the following and it should be recalled that a normed space is reflexive if and only if its dual X^* is reflexive (see, e.g., Yosida [16], p. 113).

It turns out that the properties of the duality mapping are closely related to the nature of the spaces X and X^*, more precisely to the convexity and smoothing properties of the closed balls in X and X^*.

Recall that the space X is called *strictly convex* if the unity ball B of X is strictly convex, that is the boundary ∂B contains no line segments.

The space X is said to be *uniformly convex* if for each $\varepsilon > 0$, $0 < \varepsilon < 2$, there is $\delta(\varepsilon) > 0$ such that if $\|x\| = 1$, $\|y\| = 1$, and $\|x-y\| \geq \varepsilon$, then $\|x+y\| \leq 2(1-\delta(\varepsilon))$.

Obviously, every uniformly convex space X is strictly convex. Hilbert spaces as well as the spaces $L^p(\Omega)$, $1 < p < \infty$, are uniformly convex spaces (see, e.g., Köthe [13]). Recall also that, by virtue of the Milman theorem (see, e.g., Yosida [16], p. 127), every uniformly convex Banach space X is reflexive. Conversely, it turns out that every reflexive Banach space X can be renormed such that X and X^* become strictly convex. More precisely, one has the following important result due to Asplund [4].

Theorem 1.1. *Let X be a reflexive Banach space with the norm $\|\cdot\|$. Then there is an equivalent norm $\|\cdot\|_0$ on X such that X is strictly convex in this norm and X^* is strictly convex in the dual norm $\|\cdot\|_0^*$.*

Regarding the properties of the duality mapping associated with strictly or uniformly convex Banach spaces, we have the following.

Theorem 1.2. *Let X be a Banach space. If the dual space X^* is strictly convex, then the duality mapping $J : X \to X^*$ is single-valued and demicontinuous (i.e., it is continuous from X to X_w^*). If the space X^* is uniformly convex, then J is uniformly continuous on every bounded subset of X.*

Proof. Clearly, for every $x \in X$, $J(x)$ is a closed convex subset of X^*. Because $J(x) \subset \partial B$, where B is the open ball of radius $\|x\|$ and center 0, we infer that if X^* is strictly convex, then $J(x)$ consists of a single point. Now, let $\{x_n\} \subset X$ be strongly convergent to x_0 and let x_0^* be any weak-star limit point of $\{J(x_n)\}$. (Because the unit ball of the dual space is w^*-compact (Yosida [16], p. 137) such an x_0^* exists.) We have $(x_0^*, x_0) = \|x_0\|^2 \geq \|x_0^*\|^2$ because the closed ball of radius $\|x_0\|$ in X^* is weak-star closed. Hence $\|x_0\|^2 = \|x_0^*\|^2 - (x_0^*, x_0)$. In other words, $x_0^* = J(x_0)$, and so

$$J(x_n) \rightharpoonup J(x_0),$$

as claimed. □

To prove the second part of the theorem, let us first establish the following lemma.

Lemma 1.1. *Let X be a uniformly convex Banach space. If $x_n \rightharpoonup x$ and $\limsup_{n\to\infty} \|x_n\| \le \|x\|$, then $x_n \to x$ as $n \to \infty$.*

Proof. One can assume of course that $x \neq 0$. By hypothesis, $(x^*,x_n) \to (x^*,x)$ for all $x \in X$, and so, by the weak lower semicontinuity of the norm in X,

$$\|x\| \le \liminf_{n\to\infty} \|x_n\| \le \|x\|.$$

Hence, $\lim_{n\to\infty} \|x_n\| = \|x\|$. Now, we set

$$y_n = \frac{x_n}{\|x_n\|}, \qquad y = \frac{x}{\|x\|}.$$

Clearly, $y_n \rightharpoonup y$ as $n \to \infty$. Let us assume that $y_n \not\to y$ and argue from this to a contradiction. Indeed, in this case we have a subsequence y_{n_k}, $\|y_{n_k} - y\| \ge \varepsilon$, and so there is $\delta > 0$ such that $\|y_{n_k} + y\| \le 2(1-\delta)$. Letting $n_k \to \infty$ and using once again the fact that the norm $y \to \|y\|$ is weakly lower semicontinuous, we infer that $\|y\| \le 1 - \delta$. The contradiction we have arrived at shows that the initial supposition is false. □

Proof of Theorem 1.2 (continued). Assume now that X^* is uniformly convex. We suppose that there exist subsequences $\{u_n\}, \{v_n\}$ in X such that $\|u_n\|, \|v_n\| \le M$, $\|u_n - v_n\| \to 0$ for $n \to \infty$, $\|J(u_n) - J(v_n)\| \ge \varepsilon > 0$ for all n, and argue from this to a contradiction. We set $x_n = u_n\|u_n\|^{-1}$, $y_n = v_n\|v_n\|^{-1}$. Clearly, we may assume without loss of generality that $\|u_n\| \ge \alpha > 0$ and that $\|v_n\| \ge \alpha > 0$ for all n. Then, as easily seen,

$$\|x_n - y_n\| \to 0 \quad \text{as } n \to \infty$$

and

$$(J(x_n) + J(y_n), x_n) = \|x_n\|^2 + \|y_n\|^2 + (x_n - y_n, J(y_n)) \ge 2 - \|x_n - y_n\|.$$

Hence

$$\frac{1}{2}\|J(x_n) + J(y_n)\| \ge 1 - \frac{1}{2}\|x_n - y_n\|, \qquad \forall n.$$

Inasmuch as $\|J(x_n)\| = \|J(y_n)\| = 1$ and the space X^* is uniformly convex, this implies that $\lim_{n\to\infty}(J(x_n) - J(y_n)) = 0$. On the other hand, we have

$$J(u_n) - J(v_n) = \|u_n\|(J(x_n) - J(y_n)) + (\|u_n\| - \|v_n\|)J(y_n),$$

so that $\lim_{n\to\infty}(J(u_n) - J(v_n)) = 0$ strongly in X^*. □

Now, let us give some examples of duality mappings.

1. $X = H$ is a Hilbert space identified with its own dual. Then $J = I$, the identity operator in H. If H is not identified with its dual H^*, then the duality mapping $J : H \to H^*$ is the canonical isomorphism Λ of H onto H^*. For instance, if $H = H_0^1(\Omega)$ and $H^* = H^{-1}(\Omega)$ and Ω is a bounded and open subset of $\mathbf{R}^N$, then $J = \Lambda$ is defined by

$$(\Lambda u, v) = \int_\Omega \nabla u \cdot \nabla v dx, \qquad \forall u, v \in H_0^1(\Omega). \tag{1.2}$$

In other words, $J = \Lambda$ is the Laplace operator $-\Delta$ under Dirichlet homogeneous boundary conditions in $\Omega \subset \mathbf{R}^N$. Here $H_0^1(\Omega)$ is the Sobolev space $\{u \in L^2(\Omega); \nabla u \in L^2(\Omega);\ u = 0 \text{ on } \partial\Omega\}$. (See Section 1.3 below.)

2. $X = L^p(\Omega)$, where $1 < p < \infty$ and Ω is a measurable subset of $\mathbf{R}^N$. Then, the duality mapping of X is given by

$$J(u)(x) = |u(x)|^{p-2} u(x) \|u\|_{L^p(\Omega)}^{2-p}, \quad \text{a.e. } x \in \Omega,\ \forall u \in L^p(\Omega). \tag{1.3}$$

Indeed, it is readily seen that if Φ_p is the mapping defined by the right-hand side of (1.3), we have

$$\int_\Omega \Phi_p(u) u dx = \left(\int_\Omega |u|^p dx\right)^{2/p} = \left(\int_\Omega |\Phi_p(u)|^q dx\right)^{2/q}, \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1.$$

Because the duality mapping J of $L^p(\Omega)$ is single-valued (because L^p is uniformly convex for $p > 1$) and $\Phi_p(u) \in J(u)$, we conclude that $J = \Phi_p$, as claimed. If $X = L^1(\Omega)$, then as we show later (Corollary 2.7)

$$J(u) = \{v \in L^\infty(\Omega);\ v(x) \in \operatorname{sign} u(x) \cdot \|u\|_{L^1(\Omega)},\ \text{a.e. } x \in \Omega\}. \tag{1.4}$$

3. Let X be the Sobolev space $W_0^{1,p}(\Omega)$, where $1 < p < \infty$ and Ω is a bounded and open subset of $\mathbf{R}^N$. (See Section 1.3 below.) Then,

$$J(u) = -\sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) \|u\|_{W_0^{1,p}(\Omega)}^{2-p}. \tag{1.5}$$

In other words, $J : W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega)$, $(1/p) + (1/q) = 1$, is defined by

$$(J(u), v) = \sum_{i=1}^N \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx \|u\|_{W_0^{1,p}(\Omega)}^{2-p}, \qquad \forall v \in W_0^{1,p}(\Omega). \tag{1.6}$$

We later show that the duality mapping J of the space X can be equivalently defined as the subdifferential (Gâteaux differential if X^* is strictly convex) of the function $x \to 1/2 \|x\|^2$.

1.2 Convex Functions and Subdifferentials

Here we briefly present the basic results pertaining to convex analysis in infinite-dimensional spaces. For further results and complete treatment of the subject we refer the reader to Moreau [14], Rockafellar [15], Brezis [8], Barbu and Precupanu [6] and Zălinescu [17].

Let X be a real Banach space with dual X^*. A *proper convex function* on X is a function $\varphi : X \to (-\infty, +\infty] = \overline{\mathbf{R}}$ that is not identically $+\infty$ and that satisfies the inequality

$$\varphi((1-\lambda)x+\lambda y) \leq (1-\lambda)\varphi(x)+\lambda\varphi(y) \tag{1.7}$$

for all $x, y \in X$ and all $\lambda \in [0,1]$.

The function $\varphi : X \to (-\infty, +\infty]$ is said to be *lower semicontinuous* (l.s.c.) on X if

$$\liminf_{u\to x} \varphi(u) \geq \varphi(x), \qquad \forall x \in X,$$

or, equivalently, every level subset $\{x \in X;\ \varphi(x) \leq \lambda\}$ is closed.

The function $\varphi : X \to]-\infty, +\infty]$ is said to be *weakly lower semicontinuous* if it is lower semicontinuous on the space X endowed with weak topology.

Because every level set of a convex function is convex and every closed convex set is weakly closed (this is an immediate consequence of Mazur's theorem, Yosida [16], p. 109), we may therefore conclude that a proper convex function is lower semicontinuous if and only if it is weakly lower semicontinuous.

Given a lower semicontinuous convex function $\varphi : X \to (-\infty, +\infty] = \overline{\mathbf{R}}$, $\varphi \not\equiv \infty$, we use the following notations:

$$D(\varphi) = \{x \in X;\ \varphi(x) < \infty\} \quad \text{(the effective domain of } \varphi), \tag{1.8}$$

$$\mathrm{Epi}(\varphi) = \{(x,\lambda) \in X \times \mathbf{R};\ \varphi(x) \leq \lambda\} \quad \text{(the epigraph of } \varphi). \tag{1.9}$$

It is readily seen that $\mathrm{Epi}(\varphi)$ is a closed convex subset of $X \times \mathbf{R}$, and as a matter of fact its properties are closely related to those of the function φ.

Now, let us briefly describe some elementary properties of l.s.c., convex functions.

Proposition 1.1. *Let $\varphi : X \to \overline{\mathbf{R}}$ be a proper, l.s.c., and convex function. Then φ is bounded from below by an affine function; that is there are $x_0^* \in X^*$ and $\beta \in \mathbf{R}$ such that*

$$\varphi(x) \geq (x_0^*, x) + \beta, \qquad \forall x \in X. \tag{1.10}$$

Proof. Let $E(\varphi) = \mathrm{Epi}(\varphi)$ and let $x_0 \in X$ and $r \in \mathbf{R}$ be such that $\varphi(x_0) > r$. By the classical separation theorem (see, e.g., Brezis [7]), there is a closed hyperplane $H = \{(x,\lambda) \in X \times \mathbf{R};\ -(x_0^*, x) + \lambda = \alpha\}$ that separates $E(\varphi)$ and (x_0, r). This means that

$$-(x_0^*, x) + \lambda \geq \alpha, \qquad \forall x \in E(\varphi) \quad \text{and} \quad -(x_0^*, x_0) + r < \alpha.$$

Hence, for $\lambda = \varphi(x)$, we have

$$-(x_0^*, x) + \varphi(x) \geq -(x_0^*, x_0) + r, \qquad \forall x \in X,$$

which implies (1.10). □

Proposition 1.2. *Let $\varphi : X \to \overline{\mathbf{R}}$ be a proper, convex, and l.s.c. function. Then φ is continuous on* $\operatorname{int} D(\varphi)$.

Proof. Let $x_0 \in \operatorname{int} D(\varphi)$. We prove that φ is continuous at x_0. Without loss of generality, we assume that $x_0 = 0$ and that $\varphi(0) = 0$. Because the set $\{x : \varphi(x) > -\varepsilon\}$ is open it suffices to show that $\{x : \varphi(x) < \varepsilon\}$ is a neighborhood of the origin. We set $C = \{x \in X;\ \varphi(x) \le \varepsilon\} \cap \{x \in X;\ \varphi(-x) \le \varepsilon\}$. Clearly, C is a closed balanced set of X (i.e., $\alpha x \in C$ for $|\alpha| \le 1$ and $x \in C$). Moreover, C is absorbing; that is, for every $x \in X$ there exists $\alpha > 0$ such that $\alpha x \in C$ (because the function $t \to \varphi(tx)$ is convex and finite in a neighborhood of the origin and therefore it is continuous). Because X is a Banach space, the preceding properties of C imply that it is a neighborhood of the origin, as claimed. □

The function $\varphi^* : X^* \to \overline{\mathbf{R}}$ defined by

$$\varphi^*(p) = \sup\{(p,x) - \varphi(x);\ x \in X\} \tag{1.11}$$

is called the *conjugate* of φ.

Proposition 1.3. *Let $\varphi : X \to \overline{\mathbf{R}}$ be l.s.c., convex, and proper. Then φ^* is l.s.c., convex, and proper on the space X^*.*

Proof. As supremum of a set of affine functions, φ^* is convex and l.s.c. Moreover, by Proposition 1.2 we see that $\varphi^* \not\equiv \infty$. □

Proposition 1.4. *Let $\varphi : X \to (-\infty, +\infty]$ be a weakly lower semicontinuous function such that every level set $\{x \in X;\ \varphi(x) \le \lambda\}$ is weakly compact. Then φ attains its infimum on X. In particular, if X is reflexive and φ is an l.s.c. proper convex function on X such that*

$$\lim_{\|x\| \to \infty} \varphi(x) = \infty, \tag{1.12}$$

then there exists $x_0 \in X$ such that $\varphi(x_0) = \inf\{\varphi(x);\ x \in X\}$.

Proof. Let $d = \inf\{\varphi(x);\ x \in X\}$ and let $\{x_n\} \subset X$ such that $d \le \varphi(x_n) \le d + (1/n)$. Then $\{x_n\}$ is weakly compact in X and, therefore, there is $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup x$ as $n_k \to \infty$. Because φ is weakly semicontinuous, this implies that $\varphi(x) \le d$. Hence $\varphi(x) = d$, as desired. If X is reflexive, then formula (1.12) implies that $\{x \in X;\ \varphi(x) \le \lambda\}$ are weakly compact. As seen earlier, every convex and l.s.c. function is weakly lower semicontinuous, therefore we can apply the first part. □

Given a function f from a Banach space X to $\mathbf{R}$, the mapping $f' : X \times X \to \mathbf{R}$ defined by

$$f'(x,y) = \lim_{\lambda \downarrow 0} \frac{f(x+\lambda y) - f(x)}{\lambda}, \qquad x,y \in X, \tag{1.13}$$

(if it exists) is called the *directional derivative* of f at x in direction y.

The function $f : X \to \mathbf{R}$ is said to be *Gâteaux differentiable* at $x \in X$ if there exists $\nabla f(x) \in X^*$ (the *Gâteaux differential*) such that

$$f'(x,y) = (\nabla f(x), y), \qquad \forall y \in X. \tag{1.14}$$

If the convergence in (1.13) is uniform in y on bounded subsets, then f is said to be *Fréchet differentiable* and ∇f is called the *Fréchet differential* (derivative) of f.

Given an l.s.c., convex, proper function $\varphi : X \to \overline{\mathbf{R}}$, the mapping $\partial\varphi : X \to X^*$ defined by

$$\partial\varphi(x) = \{x^* \in X^*;\ \varphi(x) \le \varphi(y) + (x^*, x-y),\ \forall y \in X\} \tag{1.15}$$

is called the *subdifferential* of φ.

In general, $\partial\varphi$ is a multivalued operator from X to X^* not everywhere defined and can be seen as a subset of $X \times X^*$.

An element $x^* \in \partial\varphi(x)$ (if any) is called a *subgradient* of φ in x. We denote as usual by $D(\partial\varphi)$ the set of all $x \in X$ for which $\partial\varphi(x) \ne \emptyset$.

Let us pause briefly to give some simple examples.

1. $\varphi(x) = 1/2\,\|x\|^2$. Then, $\partial\varphi = J$ (the duality mapping of the space X). Indeed, if $x^* \in J(x)$, then

$$(x^*, x-y) = \|x\|^2 - (x^*, y) \ge \frac{1}{2}\,(\|x\|^2 - \|y\|^2), \qquad \forall x \in X.$$

Hence $x^* \in \partial\varphi(x)$. Now, let $x^* \in \partial\varphi(x)$; that is,

$$\frac{1}{2}(\|x\|^2 - \|y\|^2) \le (x^* - y, x), \qquad \forall y \in X. \tag{1.16}$$

We take $y = \lambda x$, $0 < \lambda < 1$, in (1.16), getting

$$(x^*, x) \ge \frac{1}{2}\,\|x\|^2(1+\lambda).$$

Hence, $(x^*, x) \ge \|x\|^2$. If $y = \lambda x$, where $\lambda > 1$, we get that $(x^*, x) \le \|x\|^2$. Hence, $(x^*, x) = \|x\|^2$ and $\|x^*\| \ge \|x\|$. On the other hand, taking $y = x + \lambda u$ in (1.16), where $\lambda > 0$ and u is arbitrary in X, we get

$$\lambda(x^*, u) \le \frac{1}{2}\,(\|x + \lambda u\|^2 - \|x\|^2),$$

which yields

$$(x^*, u) \le \|x\|\ \|u\|.$$

Hence, $\|x^*\| \le \|x\|$. We have therefore proven that $(x^*, x) = \|x\|^2 = \|x^*\|^2$ as claimed.

2. Let K be a closed convex subset of X. The function $I_K : X \to \overline{\mathbf{R}}$ defined by

$$I_K(x) = \begin{cases} 0, & \text{if } x \in K, \\ +\infty, & \text{if } x \notin K, \end{cases} \tag{1.17}$$

is called the *indicator function* of K, and its dual function H,

$$H_K(p) = \sup\{(p,u);\ u \in K\}, \qquad \forall p \in X^*,$$

is called the *support function* of K. It is readily seen that $D(\partial I_K) = K$, $\partial I_K(x) = 0$ for $x \in \operatorname{int} K$ (if nonempty) and that

$$\partial I_K(x) = N_K(x) = \{x^* \in X^*;\ (x^*, x-u) \geq 0,\ \forall u \in K\}, \qquad \forall x \in K. \quad (1.18)$$

For every $x \in \partial K$ (the boundary of K), $N_K(x)$ is the *normal cone* at K in x.

3. Let φ be convex and Gâteaux differentiable at x. Then $\partial\varphi(x) = \nabla\varphi(x)$. Indeed, because φ is convex, we have

$$\varphi(x+\lambda(y-x)) \leq (1-\lambda)\varphi(x) + \lambda\varphi(y)$$

for all $x,y \in X$ and $0 \leq \lambda \leq 1$. Hence,

$$\frac{\varphi(x+\lambda(y-x)) - \varphi(x)}{\lambda} \leq \varphi(y) - \varphi(x),$$

and letting λ tend to zero, we see that $\nabla\varphi(x) \in \partial\varphi(x)$. Now, let w be an arbitrary element of $\partial\varphi(x)$. We have

$$\varphi(x) - \varphi(y) \leq (w, x-y), \qquad \forall y \in X.$$

Equivalently,

$$\frac{\varphi(x+\lambda y) - \varphi(x)}{\lambda} \geq (w,y), \qquad \forall \lambda > 0,\ y \in X,$$

and this implies that $(\nabla\varphi(x) - w, y) \geq 0$ for all $y \in X$. Hence, $w = \nabla\varphi(x)$.

By the definition of $\partial\varphi$ it is easily seen that $\varphi(x) = \inf\{\varphi(u);\ u \in X\}$ iff $0 \in \partial\varphi(x)$. There is a close relationship between $\partial\varphi$ and $\partial\varphi^*$. More precisely, we have the following.

Proposition 1.5. *Let X be a reflexive Banach space and let $\varphi : X \to \overline{\mathbf{R}}$ be an l.s.c., convex, proper function. Then the following conditions are equivalent.*

(i) $x^* \in \partial\varphi(x)$,
(ii) $\varphi(x) + \varphi^*(x^*) = (x^*, x)$,
(iii) $x \in \partial\varphi^*(x^*)$.

In particular, $\partial\varphi^* = (\partial\varphi)^{-1}$ *and* $(\varphi^*)^* = \varphi$.

Proof. By definition of φ^*, we see that

$$\varphi^*(x^*) \geq (x^*, x) - \varphi(x), \qquad \forall x \in X,$$

with equality if and only if $0 \in \partial_x(-(x^*,x) + \varphi(x))$. Hence, (i) and (ii) are equivalent. Now, if (ii) holds, then x^* is a minimum point for the function $\varphi^*(p) - (x,p)$ and so