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Fixed Points and Nonexpansive Mappings

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Fixed Points and Nonexpansive Mappings

Robert C. Sine, Editor

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PREFACE

A special session on Fixed points and Nonexpansive mappings was held in conjunction with the Annual Winter Meeting of the American Mathematical Society in Cincinnati in January 1982. This volume represents the proceedings of that meeting. In several instances the papers here go beyond "extended versions" of the actual talks given. It was felt that the interest would be greater in scope and in time if the papers took on as much of the flavor of a survey as possible.

There are other ways in which the proceedings here differ from the sessions there. Two of the original talks have been published elsewhere and therefore are not reproduced here. Now that it is known that a convex, weakly compact minimal set need not reduce to a fixed point it is natural to study what properties such a minimal set must have. Michael Edelstein addressed himself to this investigation in his talk which was published as Basic properties of nonexpansive mappings, C. R. Math Rep. Acad. Sci. Canada, vol. IV (1982) No. 2.

Barry Turett's talk demonstrated that the result due to Baillon that uniformly smooth implies the fixed point property holds, in fact, for the reason that uniformly smooth implies normal structure. Turett's result appeared as A dual view of a theorem of Baillon, in Nonlinear Analysis and Applications, S. P. Singh and O. H. Barry, Eds., Marcel Dekker, 1982.

N. Swaminathan did not speak in Cincinnati but we are fortunate to have his survey on Normal Structure included. Finally my own paper on recurrence was held in reserve at Cincinnati in the event that some scheduled participant would be unable to present his paper. This mishap did not occur so the backup paper is offered here for the first time.

My thanks to the contributors and to Ellen Swanson and her staff at the Society in Providence.

Robert Sine

Saunderstown, R.I.

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ASYMPTOTIC BEHAVIOR OF NONEXPANSIVE MAPPINGS

Ronald E. Bruck[1]

ABSTRACT. We survey what is known about the asymptotic behavior of iterates $\{T^n x\}$ of a nonlinear nonexpansive mapping defined in a Banach space, and more generally the asymptotic behavior of certain periodic evolution systems, emphasizing the nonlinear mean ergodic theorem. We also consider multiple-operator iterations, the structure of ω-limit sets, the convergence (as $t \uparrow 1$) of the often-used approximation scheme $x_t = tTx_t + (1-t)x$, and the behavior of unbounded orbits. We conclude by listing some unsolved problems and conjectures ranging from reasonable to wild-eyed.

1. INTRODUCTION. The Picard contraction-mapping principle has proved to be one of the most durable and fruitful methods in analysis. Lying just at the boundary of the Picard contractions is the set of nonexpansive mappings: those with Lipschitz constant 1. This paper is an exposition of the asymptotic behavior of nonexpansive mappings defined on convex subsets of Banach spaces.

The importance of this class lies neither in its trivial generalization of a Lipschitz condition, nor in a comparable durability or fruitfulness, but in two key observations: first, nonexpansive mappings are intimately tied to the monotonicity methods developed since the early 1960's, and constitute one of the first classes of mappings for which fixed-point results were obtained by using the fine geometric structure of the underlying Banach space instead of compactness properties. Second, they appear in applications as the transition operators for initial-value problems of differential inclusions of the form $du/dt + A(t)u(t) \ni 0$, where the operators $\{A(t)\}$ are in general multi-valued and in some sense 'positive' ('accretive', or when the sign is changed, 'dissipative'), and only minimally 'continuous'.

The asymptotic behavior of nonexpansive mappings is much richer than that of Picard contractions -- which merely converge! If T is compact, the theory of dynamical systems brings out a rich topological and algebraic structure on the ω-limit set which is quite sufficient for many applications. In this survey we emphasize results and methods using only geometry and weak compactness, since these arguments pertain so uniquely to the theory and

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maintain such a tenuous existence.

This paper is intended for a general mathematical audience. We survey those results selected as being especially elegant or surprising: asymptotic regularity, behavior of unbounded orbits, convergence methods, nonlinear mean ergodic theorems, the approximation scheme of the Abstract, and discrete approximation schemes. In the final section of the paper, titled 'Blue Sky', we present several barely tenable conjectures and indicate why they may be true.

The author regrets that innumerable fine papers have been omitted from the bibliography, even whole topics omitted from the exposition, through exigencies of time and space. He solicits corrections, comments, and bibliographical references from readers; he would especially welcome preprints and reprints bearing on the conjectures.

NOTATION, TERMINOLOGY AND STANDING CONVENTIONS. E always denotes a Banach space having norm $\|\cdot\|$, and C is a non-empty closed convex subset of E, not necessarily bounded. Unless explicitly stated otherwise, T always denotes a nonexpansive self-mapping of C, that is, a map $T : C \longrightarrow C$ such that $\|Tx-Ty\| \leq \|x-y\|$ for all x, y in C. We denote the set of fixed-points of T by F(T) (this may be empty). The set of <u>strong</u> (resp. <u>weak</u>) subsequential limits of the sequence $\{T^n x\}$ is denoted by $\omega_s(x)$ (resp. $\omega_w(x)$). The convex hull of a set S is denoted by co S, its closure by cl S, and its closed convex hull by clco S. The dual space of E is denoted by E*, and the duality pairing by $(\cdot,\cdot)$.

Since this is intended for a <u>general</u> audience, we include the definitions of some 'buzzwords'. A multivalued operator on E is a subset A of $E \times E$, i.e., a <u>relation</u> on E. We use without comment the standard set-theoretic notation A^{-1} for the inverse of A, Ax for the set of y with $[x,y] \in A$, D(A) for the effective domain of A, and R(A) for the range of A. The definitions of the sum A+B and scalar product λA are equally obvious.

The operator A is said to be <u>accretive</u> provided for each $\lambda > 0$ the operator $(I+\lambda A)^{-1}$ is nonexpansive (hence necessarily single-valued). The <u>resolvent</u> is defined by $J_\lambda = (I+\lambda A)^{-1}$, and the Yosida approximation $A_\lambda = (I-J_\lambda)/\lambda$. Thus J_λ is nonexpansive and A_λ is Lipschitzian; it follows from the contraction mapping principle that operators of the form I-T, T nonexpansive, are accretive, hence that A_λ is accretive. An accretive operator in Hilbert space is also said to be <u>monotone</u>. A <u>maximal</u> <u>monotone</u> operator on a Hilbert space E is a monotone operator which is not properly contained in another monotone operator; in that case the values Ax are closed convex sets, and the <u>minimal</u> <u>section</u> A^0 of A is defined by A^0x = the unique element of Ax of minimal norm. Maximal accretiveness is not a very useful notion in Banach spaces:, a better concept is the m-accretiveness of A: A is accretive and $R(I+\lambda A) \supset E$ for some $\lambda > 0$ (hence, by a standard argument, for all $\lambda > 0$). An accretive operator A is said to satisfy the range condition provided $R(I+\lambda A) \supset \mathrm{cl}\, D(A)$ for all sufficiently small $\lambda > 0$. The importance of the range condition is the generation theorem [59] for contraction semigroups: if A satisfies the range condition then $S(t)x = \lim_n (I+(t/n)A)^{-n}x$ exists for each x in cl D(A) and defines a contraction semigroup on cl D(A): $S(t+s) = S(t)S(s)$, S(t)x is continuous in $t \geq 0$ for each fixed x, $S(0) = I$, and each S(t) is a nonexpansive self-mapping of cl D(A).

Convex functions $f : E \longrightarrow (-\infty, +\infty]$ are permitted to assume the value $+\infty$, but not identically; f is said to be proper. It is well-known that the lower semi-continuity of such a function implies its weak sequential l.s.c. The subgradient of f at x is the set $\partial f(x)$ of w in E*, the dual space of E, which satisfy the subgradient inequality for f,

$$f(y) \geq f(x) + (w, y-x) \quad \text{for all } y \text{ in } E.$$

The subdifferential of f is the multivalued operator ∂f which assigns $\partial f(x)$ to x:, it is a generalization of the classical gradient of f. The normalized duality mapping, denoted by J, is by definition the subdifferential of $1/2\|\cdot\|^2$:, equivalently, J is characterized by

$$Jx = \{w \in E^* : \|w\| = \|x\|, \quad (w,x) = \|x\|^2\}.$$

Usually when we need the duality map, it will be in a space whose norm is Fréchet differentiable; in that case J is the actual gradient of $1/2\ \|\cdot\|^2$. For the definition of a 'weakly continuous duality map' see e.g. [35].

When E is Hilbert space and f is a proper l.s.c. convex function, ∂f is a maximal monotone operator. (This is true even when E is not a Hilbert space, but is not needed here.)

A standard reference for monotone operators and contraction semigroups in Hilbert space is Brezis [27], while Barbu [18] serves for accretive operators and contraction semigroups in Banach spaces. Browder [37] is encyclopaedic, with an extensive bibliography. It is very out of date, but the sections on accretive operators and nonexpansive mappings are still very readable. Haraux [87], [88] is essential reading for anyone interested in the periodic or almost-periodic behavior of evolution systems.

2. ASYMPTOTIC REGULARITY AND AVERAGED MAPPINGS. By an averaged mapping we mean one of the form $T_\lambda = \lambda T + (1-\lambda)I$, where $0 < \lambda < 1$ and I is the identity operator. When T is nonexpansive, so is T_λ, and both have the same fixed-point set, but T_λ has much more felicitous asymptotic behavior than the original mapping.

Krasnoselski [106] was the first to notice this regularizing effect (for $\lambda = 1/2$). He proved that if E is uniformly convex and C is compact (recall that C is ALWAYS assumed to be convex!) then the iterates of $T_{1/2}$ converge strongly to a fixed-point of T. Since the existence of a fixed-point is guaranteed by Schauder's theorem, the novelty of this result lay in its asymptotic statement. Krasnoselski gave no estimate of the rate of convergence, and it is typical of iteration methods involving nonexpansive mappings that their convergence may be arbitrarily slow; see Oblomskaja [116] for a linear example where convergence is slower than $n^{-\alpha}$ for all α in (0,1). (Patterson [125], Chapter 4 contains a thorough discussion of successive approximation methods for linear operators, and an extensive bibliography.)

Edelstein [71] observed that the uniform convexity of E in Krasnoselski's theorem can be relaxed to strict convexity, and Schaefer [152] proved convergence for a general T_λ when T is weakly continuous.

Edelstein's observation represents a recurring theme in the study of nonexpansive mappings: whereas 'static' conditions, such as strict convexity, often suffice under compactness hypotheses, 'dynamic' conditions (involving uniformity in base points) are usually needed in the absence of compactness. Thus one argument for Edelstein's version proceeds by reducing the convexity argument to points in the strong ω-limit set: the only role of strict convexity of E is to guarantee that averaged mappings are 'anti-isometric' in the sense that $\|T_\lambda x - T_\lambda y\| < \|x - y\|$ unless actually $T_\lambda x - T_\lambda y = x - y$. If C is compact then $\omega_s(x)$ -- the set of strong subsequential limits of $\{T_\lambda^n x\}$ -- is nonempty, T_λ-invariant, and T_λ is an isometry on $\omega_s(x)$. Since T_λ is anti-isometric, it acts as a translation on $\omega_s(x)$, and since $\omega_s(x)$ is bounded, each point of $\omega_s(x)$ is therefore a fixed-point of T. Thus for any y in $\omega_s(x)$ the sequence $\{\|T_\lambda^n x - y\|\}$ is decreasing (since y is a fixed-point) and has a subsequence converging to 0 (since $y \in \omega_s(x)$). This shows $T_\lambda^n x \longrightarrow y$.

Browder and Petryshyn [33] separated out the geometric condition necessary to make this work when C is not compact: T is said to be asymptotically regular on C if for each x in C, $T^n x - T^{n+1} x \longrightarrow 0$ as $n \longrightarrow \infty$. They sharpened Schaefer's result by showing that the uniform convexity of the underlying space implies that each T_λ is asymptotically regular on bounded convex sets. Kirk [96] showed that in uniformly convex spaces, a more general convex combination $\alpha_0 I + \alpha_1 T + \ldots + \alpha_n T^n$ is asymptotically regular if C is bounded and $\alpha_1 > 0$. Many papers generalized the Browder-Petryshyn theorems to iterations of the form

$$x_{n+1} = \lambda_n T x_n + (1-\lambda_n) x_n, \tag{2.1}$$

where the coefficients λ_n were assumed bounded away from 0 and 1; this is a special case of an iteration of Mann [110] (see Groetsch [83], Senter and Dotson [154], Outlaw [120], Dotson [65]).

Since these proofs seemed to use uniform convexity in an essential way, Ishikawa's 1976 discovery [93] was startling:

THEOREM 2.1. If C is bounded then T_λ is asymptotically regular for each $0 < \lambda < 1$. (There are <u>no</u> restrictions on the geometry of the Banach space!)

Edelstein and O'Brien [74] independently found a very different proof of Theorem 2.1, based on the idea of embedding E into a continuous function space and regarding $\{T_\lambda^n x\}$ as a sequence in that space. Ray and Sine [136] also give a geometric proof, based on extensions of nonexpansive mappings to an order interval in an overlying continuous-function space; but the compactness of T is essential to that method. Kirk [99] and Goebel and Kirk [81] have extended Ishikawa's method to metric spaces with convexity structure.

It has been objected that Ishikawa's proof, while self-contained, is computational and not very intuitive; but [74], [81] and [99] are also computational, and [136] is not sufficiently general. Part of the complication of Ishikawa's proof is due to his consideration of the more general iteration (2.1), where $\{\lambda_n\}$ is bounded away from 1 and $\Sigma\lambda_n = \infty$; the conclusion is that $x_n - Tx_n \longrightarrow 0$. The proof of just the asymptotic regularity of T_λ is simpler, and relies on a trivial geometric fact: if some <u>proper</u> convex combination of points on the boundary of the unit ball of a Banach space also lies on the boundary of the ball, then the points lie on a face of the ball. We need a uniform version:

LEMMA 2.1. For each $\xi > 1$, $0 < \varepsilon < 1$ there exists $\delta > 0$ (explicitly, $\delta = \varepsilon/(2\xi-1)$) such that whenever $\{v_1, v_2, \ldots, v_n\}$ are points in an annulus $S_\delta = \{u \in E : 1-\delta \leq \|u\| \leq 1+\delta\}$ of a Banach space E such that a convex combination $v^* = \sum \alpha_i v_i$ also belongs to S_δ, then every convex combination $v = \sum \lambda_i v_i$ whose coefficients satisfy the restriction $\lambda_i \leq \xi\alpha_i$ belongs to the annulus S_ε.

PROOF. Without loss of generality we may assume that each $\alpha_i > 0$ and the maximum of the ratios λ_i/α_i (which must be positive) occurs at $i = 1$; put $\beta_1 = \alpha_1/\lambda_1$, and $\beta_i = \alpha_i - \beta_1\lambda_i$ for $i > 1$. Then $\beta_1 \geq 1/\xi$, each $\beta_i \geq 0$, and $\sum \beta_i = 1$. We also have

$$v^* = \beta_1 v + \sum_{i=2}^{n} \beta_i v_i,$$

and because $v_i, v^* \in S_\delta$,

$$1 - \delta \leq \|v^*\| \leq \beta_1\|v\| + \sum_{i=2}^{n} \beta_i(1+\delta) = \beta_1\|v\| + (1-\beta_1)(1+\delta).$$

Since $\beta_1 \geq 1/\xi$, this implies $1 - \varepsilon \leq \|v\|$. Also $\|v\| \leq 1+\varepsilon$ because $\delta < \varepsilon$.

PROOF OF THEOREM 2.1. Fix λ in $(0,1)$, and put $x_n = T_\lambda^n x$, so that $x_{n+1} = \lambda T x_n + (1-\lambda)x_n$. Applying the finite-difference operator $\Delta t_n = t_{n+1} - t_n$, we obtain

(2.2) $$u_{n+1} = \lambda v_n + (1-\lambda)u_n \quad \text{for} \quad u_n := \Delta x_n, \quad v_n := \Delta T x_n.$$

Since T is nonexpansive, $\|v_n\| \leq \|u_n\|$ for all n. Thus (2.2) implies that $\{\|u_n\|\}$ is decreasing, hence convergent to a limit $d \geq 0$, and $\{\|v_n\|\}$ converges to the same limit. We may assume without loss of generality that $d > 0$.

Using (2.2) we obtain, for fixed $p > 0$, a representation of u_{n+p+1} as a convex combination

$$u_{n+p+1} = \alpha_p v_{n+p} + \alpha_{p-1} v_{n+p-1} + \dots + \alpha_1 v_{n+1} + \alpha_0 u_{n+1}.$$

All we need to know about the coefficients is that they are non-zero and independent of n. Applying Lemma 2.1 we see that for any $0 < \varepsilon < d$, for sufficiently large n <u>all</u> points in the convex hull of $\{u_{n+1}, v_{n+1}, \dots, v_{n+p}\}$ belong to the annulus $S_{d-\varepsilon}$. Taking u to be the barycenter of $\{v_{n+1}, \dots, v_{n+p}\}$ and using the definition of v_n as $\Delta T x_n$, we find in particular that $\|Tx_{n+p+1} - Tx_n\| \geq p(d-\varepsilon)$. This is impossible for ε small enough and p large enough, because $\{x_n\}$ is bounded. -Q.E.D.

A closer analysis of this proof shows that even if C is not bounded, we have still proved

$$\lim_{n\to\infty} \|T_\lambda^{n+p}x - T_\lambda^n x\| = p \lim_{n\to\infty} \|T_\lambda^{n+1}x - T_\lambda^n x\|$$

for each positive integer p. This observation and a continuous analogue were noted in [7]: if $T : C \to C$ is nonexpansive, $\delta > 0$, and $u : \mathbf{R}^+ \to C$ is a solution of the equation $du/dt + \delta(I-T)u(t) = 0$, then for each $h > 0$,

$$\lim_{t\to\infty} \|u(t+h) - u(t)\| = h \lim_{t\to\infty} \|du/dt\|.$$

For example, the Yosida approximations A_λ of an accretive operator A satisfying the range condition on $C = \text{cl}\, D(A)$ are of this form. Reich [149] subsequently generalized the results of [7] to show that for such a solution, if E is smooth and E* has a Frechet-differentiable norm, the strong $\lim_{t\to\infty} du/dt$ exists. See Section 4.

3. CONVERGENCE OF APPROXIMANTS. One of the most remarkable of the recent results on asymptotic behavior is the following theorem of Reich [147]:

THEOREM 3.1. Suppose E is uniformly smooth and C is bounded. Then for each point c of C and each $0 < t < 1$ there exists a unique point x_t in C satisfying

$$(3.1) \qquad x_t = tTx_t + (1-t)c,$$

and the strong $\lim_{t \uparrow 1} x_t$ exists and is a fixed-point of T.

This settled a problem of approximately 13 years' standing, and is the more remarkable because of its unexpectedness and the simplicity of its proof. The proof we present appeared (in slightly more general form) in Bruck-Reich [55], and also yields some insight into the nonlinear mean ergodic theorem and the notion of the asymptotic center.

The existence of a point x_t in C satisfying (3.1) is another folk theorem: $tT + (1-t)c$ is a Picard contraction of C into C (with Lipschitz constant t), and therefore has a unique fixed-point. The first version of Theorem 3.1 was proved by F. Browder [32] and B. Halpern [86] in Hilbert space; Halpern relied heavily on the inequality

$$\| x_t - x_s \|^2 \leq \| x_t - c \|^2 - \| x_s - c \|^2,$$

valid for $0 < s < t < 1$. Browder [35] partially extended this inequality to smooth spaces in the form

$$(3.2) \qquad (J(x_t - y), x_t - c) \leq 0 \quad \text{for all } y \text{ in } F(T),$$

where J is the normalized duality map of E, providing the first essential tool for the proof of Theorem 3.1. (Apply the subgradient inequality to the left side of the inequality

$$1/2 \; \| t^{-1}(x_t - c) - (y - c) \|^2 \leq 1/2 \| x_t - y \|^2.)$$

Browder used (3.2) to show that if E is uniformly convex and has a 'weakly continuous duality map' then $\{x_t\}$ is precompact; when E is merely smooth, Bruck [38, Theorem 2.3] used the weak lower semi-continuity in x_t of the left side of (3.2) to prove the uniqueness of strong subsequential limits of x_t as $t \uparrow 1$ (hence the convergence of x_t under Browder's hypotheses). See also Reich [138].

To understand Reich's idea we must first recall Edelstein's definition [72], [73] of the asymptotic center of a bounded sequence in a Banach space E. This is the set of minimizers of the (obviously convex and continuous) functional

$$(3.3) \qquad f(y) := \limsup_{n \to \infty} \| x_n - y \| .$$

One may seek to minimize f globally, or only on a certain set. Naively, one globally minimizes a convex function by setting its gradient to 0. The problem with the function defined by (3.3) is that it is not differentiable, even if the space has a very smooth norm. The problem lies in the lim sup.

Baillon [12] replaced a lim sup by a limit through a filter; when we can reduce to a separable domain, as we can in this case, we can even take the limit through a subsequence (Reich [147]). Instead, we adopt the essentially equivalent technique of Bruck-Reich [55]: we fix a Banach limit LIM (any norm-one positive linear functional on ℓ^∞ will do) and instead of (3.3) we consider

$$f(y) := \mathrm{LIM}_n \ 1/2 \ \|x_n - y\|^2. \tag{3.4}$$

Clearly f is a continuous, convex function on E and $f(y) \longrightarrow \infty$ as $\|y\| \longrightarrow \infty$. If E is uniformly smooth then f is indeed differentiable, and we compute its derivative to be

$$(f'(y),h) = -\mathrm{LIM}_n \ (J(x_n - y), h)$$

for any y and h in E. Uniformly smooth Banach spaces are reflexive, hence C is boundedly weakly compact and f assumes a minimum on C by a standard argument. Denote the set of minimizers of f on C by Min_C. Since f is smooth, each y* in Min_C satisfies the variational inequality $(f'(y^*), y-y^*) \geq 0$ for all y in C, i.e.,

$$\mathrm{LIM}_n \ (J(x_n-y^*), y-y^*) \leq 0 \quad \text{for all } y \text{ in } C.$$

Since lim inf $\leq$ LIM for any Banach limit, it follows that

$$\liminf_{n\longrightarrow\infty} (J(x_n-y^*), y-y^*) \leq 0 \quad \text{for all } y \text{ in } C, \ y^* \text{ in } \mathrm{Min}_C. \tag{3.5}$$

Now we specialize $\{x_n\}$ and prove Theorem 3.1. Let $t_n \uparrow 1$ and for each n let x_n be the unique solution of (3.1) for $t = t_n$. The nonexpansiveness of T and the monotonicity of LIM imply f is a Liapunov function for T: $f(Ty) \leq f(y)$ for all y; thus Min_C is a nonempty closed convex T-invariant subset of C. By a theorem of Baillon [13] (see Turett [160] for another aspect), bounded closed convex subsets of uniformly smooth Banach spaces have the fixed-point property for nonexpansive mappings. Thus we can choose a fixed-point y* of T in C which minimizes f on C.

On the one hand, (3.5) is satisfied for this y* because y* minimizes f. On the other, (3.2) implies

$$(J(x_n-y^*), x_n - c) \leq 0, \tag{3.6}$$

Taking y = c in (3.5) and adding (3.6), we obtain $\liminf_{n\longrightarrow\infty} \|x_n-y^*\|^2 \leq 0$.

This shows that $\{x_t : 0 < t < 1\}$ is relatively compact. The convergence of $\{x_t\}$ now follows from the uniqueness argument of Bruck [38].

For a version of Theorem 3.1 when the domain of T is not convex and T is only locally nonexpansive, see Bruck-Kirk-Reich [56].

The special property of Banach limits -- shift-invariance -- was not used in the proof of Theorem 3.1. When it is used, we obtain a very curious 'dual' mean ergodic theorem:

THEOREM 3.2. Suppose E is uniformly smooth, $T : E \longrightarrow E$ is nonexpansive, and T has a fixed-point. Then for any x in E there exists a fixed-point f of T in the asymptotic center of $\{T^n x\}$ such that the sequence $\{J(T^n x - f)\}$ is weakly almost-convergent to 0 in E^*, where J denotes the normalized duality map.

For the definition of 'weakly almost convergent' see the discussion in Section 6. The proof of Theorem 3.2 can be found in [55] (the observation that f can be taken to be in the asymptotic center of $\{T^n x\}$ is a mild variant which follows from the fact that all Banach limits yield the same value of $\mathrm{LIM}_n \|T^n x - f\|$ for f in F(T), namely the usual limit.) The requirement that T be everywhere defined is a noteworthy restriction.

4. Unbounded behavior. Another old problem which has had a recent elegant solution concerns the convergence of $\{T^n x/n\}$, when C is unbounded. This study was initiated by Pazy [126], who proved:

THEOREM 4.1. Suppose E is Hilbert and $T : E \longrightarrow E$ is nonexpansive. Then cl R(T-I) is convex and for each x in E, $\{T^n x/n\}$ converges to the point of cl R(T-I) of minimum norm.

I-T is maximal monotone when T is nonexpansive and everywhere-defined, so it was already known that cl R(T-I) is convex; but when T is only defined on a convex subset C, this set may fail to be convex, and the main obstruction in this case appeared to be whether R(I-T) has the minimum property. (A subset D of E is said to have the minimum property if dis (0, D) = dis (0, co D).) Subsequently Reich [146], [148], [149] has shown that it does (in a very wide class of Banach spaces).

Note that $\{\|T^n x/n\|\}$ always converges. This is an old semigroup trick:

LEMMA 4.1. If $\{a_n\}$ is a sequence of positive real numbers such that

$$a_{m+n} \leq a_m + a_n \quad \text{for all } m, n \in \mathbf{Z}^+, \tag{4.1}$$

then $\{a_n/n\}$ converges.

PROOF. It is a folk theorem that if a subset B of $\mathbf{Z}^+$ is closed under addition and contains two consecutive integers, then B contains all sufficiently large integers. (If $a, b \in B$ are relatively prime and $c \in \mathbf{Z}$, find integers x and y with $ax+by = c$. Adding multiples of b to x while simul-