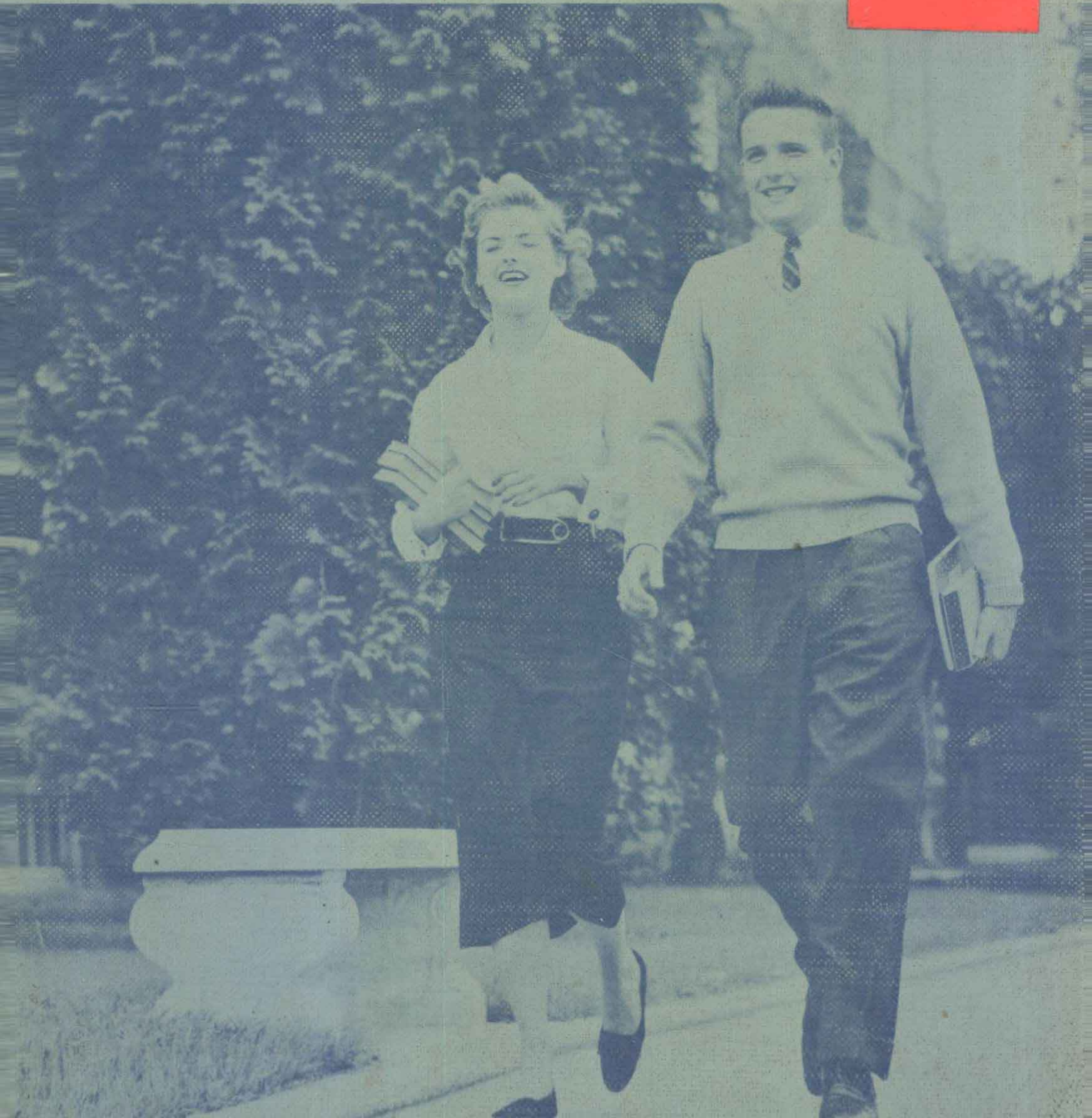
ALGEBRA

SECOND COURSE

MAYOR and WILCOX

$1.00 EACH



ALGEBRA

SECOND COURSE

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to the teacher

The study of mathematics, which has always been important, is now indispensable in any program of education. The second course in algebra is particularly relevant to the needs of today. Through such a course, students are introduced to ideas and techniques basic to the development of modern science.

In addition, the study of algebra gives access to the enjoyment of a rich heritage of knowledge. There is much beauty in mathematics, and no one can be counted truly educated who does not appreciate that fact.

During two years of algebra the student assimilates mathematical theory that is the fruit of centuries of investigation. In doing so, he not only opens areas of self-development but may, in time, apply his knowledge to the technological tasks facing our country.

Again the authors wish to express their thanks to many people who have assisted in the development of *Algebra, Second Course.* Special thanks go to the hundreds of students who have been in our classes, for their questions and for the satisfaction they have shown from their study of algebra, always a source of help and inspiration to any teacher. We are also deeply grateful to the hundreds of teachers in all parts of the country with whom we had the privilege of discussing the many problems which face a teacher of mathematics.

We owe much to William B. Wilcox and Darlene Mayor for their valuable assistance in all stages of the preparation of this manuscript. The constructive suggestions of Ruby Wells, New Albany High School, New Albany, Indiana; John A. Brown, State University Teachers College, Oneonta, New York; and Joseph W. Kennedy, Wisconsin High School, Madison, Wisconsin, have also been very helpful. We wish to thank the editorial staff of Prentice-Hall, Inc., and in particular Charles D. Smith and Lillian Margot, for their patience, encouragement, and competent aid in the publication of both books, *Algebra, First Course* and *Algebra, Second Course.* The photo reproduced on the cover is by Ewing Galloway.

contents

1 coordinates of points on a line

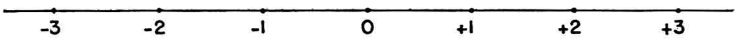
1.1. Integers and Points. Engineers, scientists, statisticians, and members of many other professions will tell you that they use mathematics in their work. They seldom say that they use algebra or geometry—they say, "mathematics." As a matter of fact, the branches of mathematics are interrelated. When you studied your first course in algebra, you found that you used your knowledge of arithmetic daily. You solved exercises about geometric figures, and you were concerned with geometric properties. You had an introduction to numerical trigonometry.

Let us consider the fundamental elements of arithmetic, algebra, and geometry. Numbers are elements of arithmetic and algebra, and points are elements of geometry. Suppose we draw a line and designate points at equal distances on the line by the whole numbers 1, 2, 3, 4, and so on. These numbers are called the *natural numbers*. You have seen numbers used in this manner on a ruler.

We have now set up a one-to-one correspondence between certain points on the line and the natural numbers. To locate one of these points on the line, we need merely to know a number. This number is called the *coordinate* of the point.

We could extend our line indefinitely to the right. There would be no limit to the number of points such as those we have already marked, and no limit to the number of natural numbers with which we could designate them. We can say that there is an infinite number of each.

If we extend our line to the left, we need to designate a beginning point. We shall call it *zero*. Then, to show that we are moving in the opposite direction from zero, we shall use negative numbers to designate the points through which we pass. We have now expanded our idea of number to include negative numbers.



This is one way in which we can see that algebra is an extension of arithmetic. Remembering that we are numbering points on the line, we may say that we are also studying geometry. As a matter of fact, we are studying mathematics. In this book, we shall study mathematics with emphasis on algebra.

The numbers which we used to designate points on the line drawn above were all whole numbers. These numbers are called *integers*. Let us find the distance between two points on our line. The distance may be expressed as the number of units between the points. In this case, we would be giving the absolute value of the length of the segment. We could also state this distance as a positive or negative number. Our number would then show the direction, as well as the distance, from one point to the second. The absolute value of both $+3$ and -3 is 3, written $|3|$. The distance

from point $A(-3)$ to point $B(+2)$ is 5 units. In moving from A to B, we move in the positive direction. As a directed line segment, $AB = +5$. Moving from B to A, we would read the segment as BA, and $BA = -5$. However, $|AB| = |BA| = 5$.

Example

Find the directed distance CD. $|CD| = 4$. $CD = -4$.

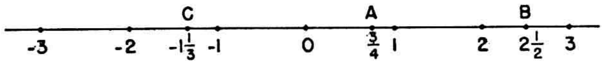
EXERCISES

Draw a number scale from -8 to $+8$. Locate the points listed below. Find the absolute value of the distance between the points indicated. Find the value of the distance as a directed line segment.

1. $A(+3)$, $B(-2)$. AB
2. $C(-7)$, $D(0)$. DC
3. $E(+6)$, $F(+2)$. EF
4. $G(-1)$, $H(+7)$. GH
5. $J(+1)$, $K(+4)$. JK
6. $L(-8)$, $M(-6)$. LM
7. $N(-3)$, $O(-4)$. NO
8. $R(0)$, $S(-4)$. SR
9. $W(-7)$, $X(+3)$. WX
10. Could you have used algebraic addition or subtraction to find your answers for Problems 1 through 9? Explain.

◆11. If the coordinate of Y is (x_1) and of Z is (x_2), how could you represent the directed distance YZ? ZY?

1.2. Rational Numbers. Just as there are points between the points which we have numbered on the line in article 1, there exist numbers with which to designate these points. To obtain these numbers, we first need to expand our knowledge of number to include fractions. This you did many years ago in arithmetic. You associated these numbers with points on a line by using halves, fourths, eighths, and other fractions of an inch on a ruler. Below, we have points $A(+\frac{3}{4})$, $B(+2\frac{1}{2})$, and $C(-1\frac{1}{3})$.



The integers and fractions of the type we have used for the coordinates of A, B, and C are called *rational numbers.*

> A rational number is a number which may be expressed as the ratio of two integers.

Three is a rational number because it can be written $\frac{3}{1}$; $2\frac{2}{5}$ is a rational number because it can be written $\frac{12}{5}$.

EXERCISES

1. Tell which of the following numbers are integers: (a) 1.5 (b) $\frac{2}{3}$ (c) -117 (d) $-\frac{6}{5}$ (e) 0 (f) $+7$ (g) -3
2. Show that each of the following numbers can be written as the ratio of two integers: (a) $1\frac{1}{2}$ (b) 7 (c) $8\frac{2}{3}$ (d) 1.25 (e) $\frac{5}{6}$ (f) $21\frac{1}{5}$
3. Find the directed distances AB, CD, EF, MN, RS, XY, and WZ if the coordinates of these points are: $A(-2\frac{1}{2})$, $B(+1)$, $C(-1\frac{1}{3})$, $D(-3\frac{1}{2})$, $E(-\frac{1}{2})$, $F(+6)$, $M(+10\frac{1}{2})$, $N(-2\frac{1}{2})$, $R(-1\frac{1}{2})$, $S(+1\frac{1}{2})$, $W(-\frac{3}{4})$, $X(0)$, $Y(-1.4)$, and $Z(\frac{3}{4})$.

1.3. Irrational Numbers. If we were to establish a one-to-one correspondence between all rational numbers and points on our line, there would be some additional points for which we would not have corresponding numbers according to our discussion thus far. The numbers we use for these points are called *irrational numbers.* An irrational number cannot be written as the ratio of two integers. $\sqrt[3]{3}$, $2\sqrt{6}$, and π are examples of irrational numbers.

Before we study irrational numbers further, we need to review such basic ideas as: variables, equations, powers, roots, and the Pythagorean Theorem.

The *real* numbers include all rational and irrational numbers.

1.4. Variables and Equations. If we wish to represent the coordinate of a point on our line by a number but do not intend to represent the coordinate of a particular point, we may represent this coordinate by a letter, such as x. Used in this way, x is a symbol for which we may substitute any one of a collection of numbers, and it is called a *variable.* The collection in this case contains all the real numbers. Such a collection may be called a *set.* We shall learn more about variables and sets in later chapters of this book.

Algebraic expressions contain numbers and letters for which numbers may be substituted. Examples of algebraic expressions are: $2x$, $a + 3$, r^2s, $\frac{2x}{3}$, 5, $\frac{x - 9}{7}$, $x^3y + xy^3 - y^4$. Algebraic expressions like $2x$, 7, x^3t, and $\frac{x}{y}$ are called *terms.*

> An equation is the statement of the equality of two algebraic expressions.

Let us consider equations containing one unknown, such as: $x + 5 = 21$, $3x + 1 = 11 - 2x$, $x^2 - 6x = 7$, and $\frac{4x}{3} = 28$.

An equation that is true for all values of the unknown is called an *identical equation* or, simply, an *identity*. An equation that is true only for a definite number of values of the unknown is called a *conditional equation*. We shall usually speak of a conditional equation as an *equation*.

In a conditional equation, a value for the unknown number which, when substituted for the unknown in the equation, will change the equation to an identity is called a *root* of the equation. We say that this value for the unknown *satisfies* the equation.

A root of the equation $3x - 4 = 2x + 6$ is $+10$.

$$\begin{array}{rcl} 3x - 4 & = & 2x + 6 \\ 3(+10) - 4 & & 2(+10) + 6 \\ 30 - 4 & & 20 + 6 \\ 26 & = & 26 \end{array}$$

Example

Is -2 a root of the equation $2(x - 1) - 3x = x + 8$?

$$\begin{array}{rcl} 2(x - 1) - 3x & = & x + 8 \\ 2(-3) + 6 & & -2 + 8 \\ -6 + 6 & & +6 \\ 0 & \neq & +6 \end{array}$$

($\neq$ is read "does not equal.")

Since our equation does not become an identity when -2 is substituted for x, -2 is not a root of the equation.

EXERCISES

Determine whether or not the number following the equation is a root of that equation. Make your decision by substituting the number for the unknown in the equation.

1. $x - 5 = 7.$ (12)
2. $3x - 1 = 4x.$ (1)
3. $14 - 3b = 2 + b.$ (3)
4. $5y + 2 - 3y = 11.$ (5)
5. $4x + 5 = x.$ (-1)
6. $6a + 5 = 8 - 3a.$ $(\frac{1}{3})$
7. $2x + 25 = 4 - x.$ (-3)
8. $2(6x - 1) = 11x + 5.$ (7)
9. $3(1 - 2x) = 4(x + 12).$ $(-4\frac{1}{2})$
10. $2(x - 8) - 3(4 + x) = -27.$ (4)
11. $11 - 6w = 16 - 2w.$ $(-1\frac{1}{4})$
12. $2r - 7r = 15 + r.$ $(3\frac{3}{4})$
13. $5.1 - (3x - 1) = 13 - 6x.$ (2.3)
14. $2(x - .7) = 2.5 - x.$ (1.1)
15. $1.4y + 9 = 2.6 + .6y.$ (2)
16. $\frac{z}{2} - \frac{z}{3} = 1.$ (6)
17. $\frac{3x}{4} - \frac{x}{5} = \frac{33}{10}.$ (-5)
18. $\frac{2x - 5}{3} = \frac{x - 2}{2}.$ (4)
19. $\frac{1 - 4x}{3} = 4\frac{1}{3}.$ (3)
20. $\frac{5 + x}{2} + \frac{x + 3}{5} = -\frac{2}{5}.$ (-3)
21. $\frac{7}{3x} - \frac{5}{2x} = -\frac{2}{5}.$ (1)
22. $\frac{3}{x - 5} = \frac{5}{x - 1}.$ (11)
23. $\frac{2}{2x + 3} = \frac{12}{x - 4}.$ (-2)

◆24. $\frac{2x}{3} - \frac{5x}{4} = \frac{7}{12}.$ $\left(\frac{1}{3}\right)$

◆25. $\frac{x + 5}{5} = \frac{2x + 1}{4} + \frac{x + 5}{5}.$ $\left(\frac{2}{5}\right)$

◆Which of the following equations are conditional equations, and which are identical equations?

26. $x + 3 = 2x$
27. $5 - x = 5 - x$
28. $3x - 7 - 2x = 10$
29. $3(x + 1) - 5 = 8 - 3x$
30. $4x + 3 - 2x = 3 + 2x$
31. $4 = 4$
32. $5(x - 2) - (3x - 1) = 2(x - 4) - 1$
33. $\frac{x}{3} - \frac{2}{5} = \frac{5x - 6}{15}$
34. $\frac{6x}{7} - \frac{1}{2} = \frac{5}{14}$
35. $\frac{x + 4}{6} = \frac{x + 2}{3}$

1.5. Solution of Equations. Solving an equation means finding the root or roots of the equation.

Simple equations may be solved by the use of addition, multiplication, and division. Mathematicians use the following rules in solving equations:

1. If the same number is added to both members of an equation, the sums are equal.
2. If both members of an equation are multiplied by the same number, the products are equal.
3. If both members of an equation are divided by the same number, the quotients are equal.

EXAMPLES

1. Solve and check: $5x - 2 = 8 - 3x$.

$$\begin{aligned} 5x - 2 &= 8 - 3x \\ 8x &= 10 \\ x &= 1\tfrac{1}{4} \end{aligned}$$

Check:
$$\begin{array}{rcl} 5x - 2 &=& 8 - 3x \\ 6\tfrac{1}{4} - 2 && 8 - 3\tfrac{3}{4} \\ 4\tfrac{1}{4} &=& 4\tfrac{1}{4} \end{array}$$

2. Solve and check: $2(3x - 5) - 4x = 2 - x$.

$$\begin{aligned} 2(3x - 5) - 4x &= 2 - x \\ 6x - 10 - 4x &= 2 - x \\ 6x - 4x + x &= 2 + 10 \\ 3x &= 12 \\ x &= 4 \end{aligned}$$

Check:
$$\begin{array}{rcl} 2(3x - 5) - 4x &=& 2 - x \\ 2(7) - 16 && 2 - 4 \\ 14 - 16 && 2 - 4 \\ -2 &=& -2 \end{array}$$

3. Solve and check: $\dfrac{2x - 5}{3} - \dfrac{4 - x}{6} = \dfrac{3x}{12}$

$$\begin{aligned} \frac{2x - 5}{3} - \frac{4 - x}{6} &= \frac{3x}{12} \\ 12\left(\frac{2x - 5}{3}\right) - 12\left(\frac{4 - x}{6}\right) &= 12\left(\frac{3x}{12}\right) \\ 4(2x - 5) - 2(4 - x) &= 1(3x) \\ 8x - 20 - 8 + 2x &= 3x \\ 7x &= 28 \\ x &= 4 \end{aligned}$$

Check:
$$\begin{array}{rcl} \dfrac{2x - 5}{3} - \dfrac{4 - x}{6} &=& \dfrac{3x}{12} \\ \dfrac{8 - 5}{3} - \dfrac{0}{6} && \dfrac{12}{12} \\ \dfrac{3}{3} - 0 && 1 \\ 1 &=& 1 \end{array}$$

EXERCISES

Solve and check:

1. $2x - 3 = 7x + 12$
2. $7 - 4y = 15 - 2y$
3. $7b - 4 + 2b = 32$
4. $6 + 8x - 2 = 20$
5. $11x = 7x + 2$
6. $3w = 9 - 15w$
7. $1 - 4x = 11$
8. $6y - 5 = 10$
9. $2(a - 3) = 16$
10. $5(3 - 2c) = 25$
11. $4x + 2(x + 5) = 8$
12. $9 + 7(2x - 1) = -26$
13. $3(2y + 5) = 5(y - 1)$
14. $7(1 - 3m) = 2(2m - 9)$
15. $11 - 2(r + 7) = 0$
16. $21 = 8 - 5(2x + 3)$
17. $3(4 + 11y) = -1 - 2(9 - y)$
18. $5(2x - 5) + 9 = 4(7 - 3x)$
19. $11 - (2x - 1) = x$
20. $5x = 18 - (3 - 2x)$
21. $4(x + 7) - 3(2x - 1) = 7$
22. $6(3x + 5) = 5(4x - 3) + 33$
23. $23 - (x + 7) = 3(4 + x)$
24. $8x - 5(3x - 4) = 3(x - 10)$
25. $3.2x - 7.5 = 5.3$
26. $4.7x + 7.2 = 3.1x$
27. $3x - 1.7 = 7$
28. $13.8 - 3.6x = 2.4x$
29. $1.5x - 8 = 1.7x$
30. $2.3y = 21 + 1.6y$
31. $2(x - .5) = 5(3.4 - 1.2x) + 2x$
32. $1.2(2x - 6) = 3.6x$
33. $\frac{x}{2} - \frac{3x}{5} = \frac{7}{5}$
34. $\frac{x}{2} + \frac{2x}{3} = \frac{14}{3}$
35. $\frac{x}{2} - \frac{5}{2} = \frac{x}{8}$
36. $\frac{3x}{10} - \frac{x}{5} = \frac{5}{2}$
37. $\frac{x - 3}{5} = 7$
38. $\frac{3x + 5}{4} = -1$
39. $\frac{2x + 10}{5} = \frac{4 - 3x}{2}$
40. $\frac{5 - 8x}{3} = \frac{2x - 7}{5}$
41. $\frac{x + 3}{2} - \frac{2x + 5}{3} = 5$
42. $\frac{5x - 3}{4} - \frac{2x + 1}{3} = 3$

Solve for x or y in Problems 43–50:

◆43. $ax + b = c$

◆44. $b + cx = m$

◆45. $2(x + a) = 5a$

◆46. $4x - 3(x - b) = 8b$

◆47. $\frac{a}{2} - \frac{y}{3} = 5$

◆48. $\frac{x}{2} - \frac{b}{3} = \frac{c}{6}$

◆49. $\frac{x - m}{2} - \frac{2x}{5} = 3m$

◆50. $\frac{3x - c}{4} - \frac{x + 5c}{2} = c$

51 through 75. Solve and check equations 1 through 25 in article 1.4 of this chapter.

1.6. Powers and Roots. If we multiply a number by itself one or more times, we say that we are raising it to a power. Since $2 \times 2 \times 2 = 8$, 8 is the third power of 2. The 2 is used three times as a factor. We may also write $(2)^3 = 8$. In this statement, 3 is said to be used as an *exponent*. In the expression, $(2)^3 = 8$, 2 is the base and 3 is the exponent. x^4 means $(x)(x)(x)(x)$; a^3b^2 means $(a)(a)(a)(b)(b)$.

Subtraction is the process inverse to addition. Thus, $3 + 2 - 2 = 3$. Division is the process inverse to multiplication: $\frac{3(2)}{2} = 3$. The symbol $\sqrt{}$, which is called a *radical* sign, indicates that a root is to be taken. Taking a root is the process inverse to raising to a power. Since $(2)^3 = 8$, $\sqrt[3]{8} = 2$. Also, $(3)^2 = 9$ and $\sqrt{9} = 3$. The square root of 9 could also be -3. When we write $\sqrt{9}$, we shall mean the positive square root; $-\sqrt{9}$ will indicate the negative square root.

The square root of a number is one of two equal factors the product of which is the number.

Examples

Find: (a) $\sqrt{(2x)^2}$ (b) $\sqrt{25}$ (c) 5^3 (d) $(2^3)^2$ (e) $(X^2)^3$

(a) $\sqrt{(2x)^2} = 2x$ (b) $\sqrt{25} = \sqrt{5^2} = 5$ (c) $5^3 = (5)(5)(5) = 125$
(d) $(2^3)^2 = (2^3)(2^3) = (8)(8) = 64$ (e) $(X^2)^3 = (X^2)(X^2)(X^2) = X^6$

EXERCISES

Find the value of each of the following:

1. 4^2
2. 3^3
3. 1^5
4. 5^4
5. $2(12)^2$
6. $(2^2)^3$
7. $(2)^2(3)^2$
8. $(2 \cdot 3)^2$
9. $(3)^2(2)^3$
10. $(10)^6$
11. $3(2)^3$
12. $4(5)^2$

13. $\sqrt{4}$	14. $\sqrt{(3)^2}$	15. $\sqrt{5^2}$	16. $\sqrt{81}$
17. $2\sqrt{25}$	18. $3\sqrt{49}$	19. $\sqrt{1}$	20. $\sqrt{100}$
21. $6\sqrt{4}$	22. $8\sqrt{9}$	23. $10\sqrt{36}$	24. $(2^2)\sqrt{16}$

Find the squares or square roots, as indicated:

25. $\sqrt{x^2}$	26. $\sqrt{y^4}$	27. $\sqrt{x^2y^4}$	28. $\sqrt{9r^2}$
29. $\sqrt{36c^4}$	30. $\sqrt{36y^2}$	31. $(3x)^2$	32. $3(x)^2$
33. $(xy)^2$	34. $2(xy)^2$	35. $(2xy)^2$	36. $(3ab^3)^2$
37. $3(ab)^3$	38. $3a(b^3)^2$	39. $(x^2y^3)^2$	40. $(-2)^2$
41. $(-2x)^2$	42. $(-2x^2)^2$		

1.7. Simplifying Radical Expressions. In elementary algebra, you learned to simplify some radical expressions. The methods you used were based on the principles that $\sqrt{xy} = (\sqrt{x})(\sqrt{y})$ and $\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$. We shall understand these two principles better if we substitute for x and y numbers which are perfect squares.

$$\sqrt{36} = (\sqrt{9})(\sqrt{4}) = (3)(2) = 6$$

$$\sqrt{\frac{36}{4}} = \frac{\sqrt{36}}{\sqrt{4}} = \frac{6}{2} = 3$$

Examples

Simplify: (a) $\sqrt{50}$ (b) $\sqrt{8x^2y^3}$ (c) $\sqrt{\frac{3}{8}}$ (d) $\sqrt{\frac{4}{5}}$

(a) $\sqrt{50} = \sqrt{(25)(2)} = 5\sqrt{2}$

(b) $\sqrt{8x^2y^3} = \sqrt{(4x^2y^2)(2y)} = 2xy\sqrt{2y}$

(c) $\sqrt{\frac{3}{8}} = \sqrt{\frac{6}{16}} = \frac{\sqrt{6}}{\sqrt{16}} = \frac{\sqrt{6}}{4}$ or $\frac{1}{4}\sqrt{6}$

(d) $\sqrt{\frac{4}{5}} = \sqrt{\frac{4(5)}{25}} = \frac{2\sqrt{5}}{5} = \frac{2}{5}\sqrt{5}$

EXERCISES

Simplify:

1. $\sqrt{8}$	2. $\sqrt{75}$	3. $\sqrt{20}$	4. $\sqrt{32}$
5. $\sqrt{500}$	6. $\sqrt{162}$	7. $\sqrt{125}$	8. $\sqrt{320}$
9. $\sqrt{60}$	10. $\sqrt{90}$	11. $\sqrt{72}$	12. $\sqrt{44}$