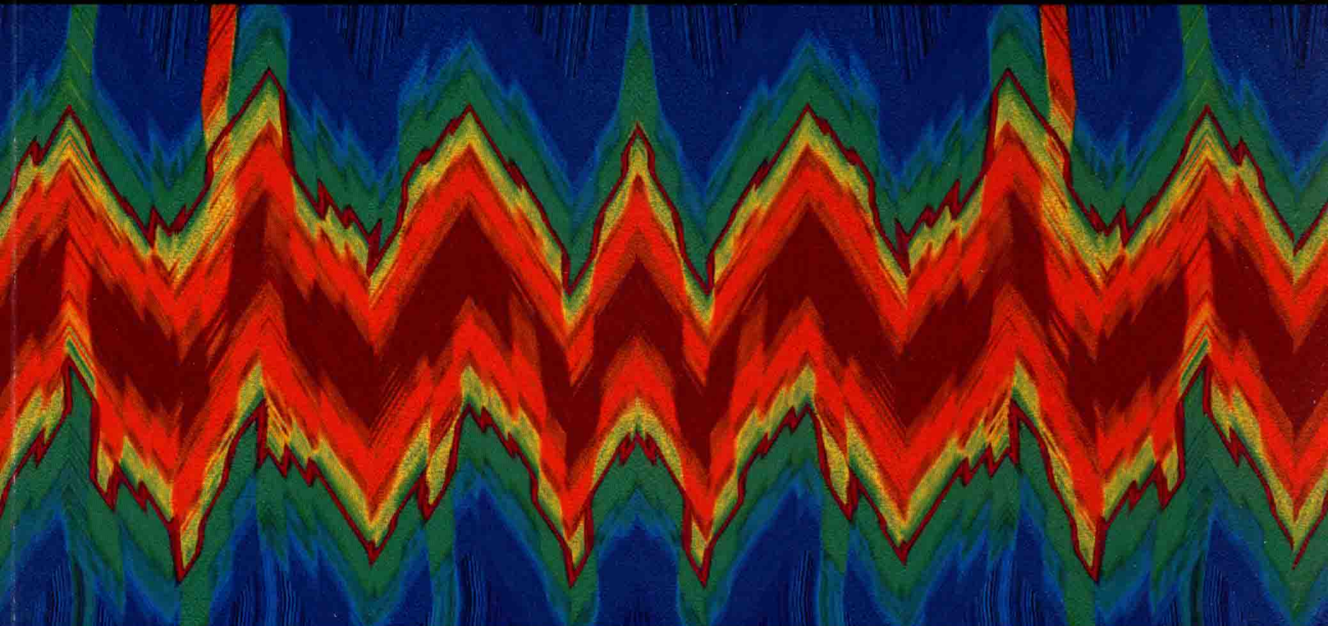
Wavelet Analysis

Lizhi Cheng • Hongxia Wang • Yong Luo • Bo Chen



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Preface

This book is written according to the lecture notes of the graduate course "wavelet analysis with its applications". The first part of the course is for graduate students in applied mathematics and the second part is for electrical engineering Ph. D students. It can be used either as the textbook for a course that focuses on wavelets, or as reference book for a larger course in harmonic analysis or signal processing.

From the view of mathematics, wavelet analysis is to represent or approximate functions by a special class of basis called wavelets. As a fast and efficient approximation method with high precision, wavelet theory is an important development of Fourier analysis in the field of harmonic analysis. Different from trigonometric functions used by Fourier transform, the wavelets are fast vanished or compact supported, which makes it possible for local analysis both on time and frequency domain. Furthermore, the multiscale resolution analysis (MRA) of wavelets is very crucial in the processing of non-stationary signal.

On the other hand, classical wavelet theory also has many limitations in application even though it has made great success in many fields especially in image processing. So some new progress of wavelet theory is also included in this book. For example, the lifting scheme proposed by W. Sweldens to accelerate the wavelet transform and obtain non-lossless information representation is introduced at the end of the first part. Besides, to derive shift invariable transform with less redundancy, N. Kingsbury proposed a dual tree complex wavelet (DTCW) transform which is briefly described in chapter 8 together with image denoising. Furthermore, in order to get sparse representation for multi-dimensional signal with low-dimensional features, the well- known ridgelet and curvelet theory is introduced in chapter 9 as an important multiscale geometric analysis method.

As this book is intended as a graduate-level textbook both for engineering and applied mathematics, in addition to the basic introduction of classic wavelet theory, we try our best to introduce the latest advancements of MRA both in theory and applications. Apart from DTCW, ridgelet and curvelet mentioned above, the design of M-band wavelets and integer transforms are also introduced in this book together with their applications in image compression and digital watermark.

This book is divided into two parts. The fundamentals of wavelets theory are briefly introduced in Part I which involves six chapters. We focus on the applications of wavelets in Part II.

Chapter 1 is the fundamentals of Fourier theory which includes Fourier series, the theories and algorithms for continuous and discrete Fourier transforms. The necessary mathematical concepts related to wavelets are introduced in chapter 2. In chapter 3 we describe the Haar functions in detail. It is the starting point to understand the MRA of wavelets. Whereas chapter 4 is dedicated to MRA theory together with how to construct wavelet functions. chapter 5 is mainly about the design of M-band wavelets and multi-wavelets. We introduce the concepts of filter banks, QMF together with their construction. In the last chapter of Part I, the lifting scheme of wavelet transform is included in chapter 6.

The topic of chapter 7 is image compression based on wavelet transform. The encoding and decoding algorithm for wavelet coefficients are described at full length in this chapter. The wavelet based image denoising and enhancement is introduced in chapter 8. Based on the introduction of several typical algorithms, the DTCW based image denoising and enhancing methods are described. The theory and algorithm of ridgelet & curvelet transform is given in charpter 9. As a useful application of wavelets, digital watermark technique in wavelet domain is introduced in charpter 10. As the last two chapters of this book, how to solve PDE and linear system based on wavelet is given in chapter 11 and 12.

We wish this book provide a bridge between mathematical theory and engineering applications. It is still challenging even though many books about wavelets have been published these years. Most of the materials and resources adopted in this book are coming from our research work recent ten years in the theory and method of signal processing supported by National Natural Science Foundation of China, National High-tech R&D Program of China (863 Program) and "9th Five-Year, 10th Five-Year, 11th Five-Year" Defense Research Foundation of China. The viewpoint taken in the presentation of the material is of course highly subjective and a bias towards our own research is obvious. Nevertheless, we hope that the book will stimulate the interest of students and researchers in the field. Look forward to all the criticism and suggestions which are so valued to improve the book.

This book would not have been possible without the help of our friends and colleagues who have made many valuable comments and spotted numerous mistakes. In particular, we would like to thank Hanwei Guo, Zenghui Zhang, Hongping Cai, Xiongming Zhang, Chuanhua Shu, Hui Zhang, Housen Li and Fengxia Yan for a careful reading of the text. We owe special thanks to Diannong Liang and Zhengming Wang who encouraged us to write this book. Even though the scope of the text has changed over the years, the original initiative is due to them. Finally, we thank all the students who attended tutorials or classes taught by us and whose critical comments helped us to organize our thoughts.

Lizhi Cheng, Hongxia Wang
Changsha, June 2014

Contents

Chapter 1

Overview of Fourier Analysis

1.1 Introduction

Inspired by mathematical model of thermal diffusion, Fourier, the famous scientist, in a report presented to the French National Academy of Science in 1807, pointed out that any periodic function can be expressed by a series of sinusoid. Through the improvements and developments for a century and a half, the harmonic analysis theory with the main research contents of Fourier series and Fourier integral has been widely used in mathematics, physics and engineering practice. Wavelet theory which will be especially focused on is just established after thoroughly studying the characteristics and limits of Fourier analysis method. So, combining the needs of this book, basic concept and theory are focused on in this chapter.

Firstly, the following examples demonstrate that Fourier analysis is widely applied in practice.

Example 1.1 Consider the following equation of heat conduction

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), & 0 < t, 0 \leqslant x \leqslant \pi \\ u(x,0) = f(x), & 0 \leqslant x \leqslant \pi \\ u(0,t) = 0, & u(\pi,t) = 0 \end{cases} \tag{1.1}$$

The solution $u(x,t)$ of the differential equation above describes the temperature of a conductor in the position of x at time t. When $t = 0$, the initial temperature at x is $f(x)$. When $x = 0, x = \pi$, its temperature remains unchanged.

Then consider the use of separation of variables to solve the equation above. Suppose the solution has the following form

$$u(x,t) = Y(x)V(t)$$

Y is a function of x with its domain $0 \leqslant x \leqslant \pi$ and V of t with its domain $0 \leqslant t$.

Substituting the above expression into (1.1) directly we get

$$Y(x)V'(t) = Y''(x)V(t), \text{ or equivalent equation } \frac{V'(t)}{V(t)} = \frac{Y''(x)}{Y(x)} \tag{1.2}$$

The left-hand side of the second expression in (1.2) is a function of t and the right-hand side is a function of x. Because t and x are independent with each other, there exists constant c satisfying $\frac{V'(t)}{V(t)} = \frac{Y''(x)}{Y(x)} = c$. According to the first expression of equations above, we get $V(t) = de^{ct}$ is for some constant d. Considering the physical property of $V(t)$, the constant c cannot be a positive number (otherwise $\lim_{t\to+\infty} |V(t)| = +\infty$), meanwhile c cannot be 0 (otherwise $V(t) = d$), so there exists a positive number λ with $c = -\lambda^2$, that is $V(t) = de^{-\lambda^2 t}$. Substituting it into (1.2), we get

$$Y''(x) + \lambda^2 Y(x) = 0, \quad 0 \leqslant x \leqslant \pi, \quad Y(0) = 0, \quad Y(\pi) = 0 \tag{1.3}$$

Solving the boundary value problem of differential equation, we obtain

$$Y(x) = a\cos(\lambda x) + b\sin(\lambda x)$$

By the boundary condition, we have $a = Y(0) = 0$ and $Y(\pi) = b\sin(\lambda\pi) = 0$, knowing λ must be positive integer, let $\lambda = k$, and the corresponding coefficient b changes to b_k, that is to say, $Y_k(x) = b_k \sin(kx)$ is the solution to (1.3). Noting that for any natural number k, $Y_k(x)$ satisfies (1.3), combining $V(t) = de^{-\lambda^2 t}$, we obtain

$$u_k(x,t) = b_k e^{-k^2 t}\sin(kx)$$

a solution to (1.1), while general solutions can be obtained by superimposing particular solutions

$$u(x,t) = \sum_{k=1}^{+\infty} b_k e^{-k^2 t}\sin(kx) \tag{1.4}$$

By $f(x) = u(x,0)$ and (1.4), we have

$$f(x) = \sum_{k=1}^{+\infty} b_k \sin(kx) \tag{1.5}$$

(1.5) is called the Fourier series expansion of $f(x)$. And coefficients b_k are determined by$f(x)$. As for the details, it will be discussed in the following sections.

Example 1.2 Signal analyzing method on the basis of Fourier series.

Investigating the sinusoid function $\sin(kt)$, obviously, its period is $2\pi/k$ and the corresponding frequency is k. Sounds generated by general musical instruments and signals like voltage can be expressed by the sum of sinusoid function with different frequencies. For instance, the signal $100\sin(t) + 3\sin(20t) - 0.5\sin(100t)$ vibrates 1, 20 and 100 times respectively in a time period of 2π, in which, the amplitude of 1-frequency component is the largest, reaching 100 (having the decisive effect). Generally, signal $f(t)$ can be decomposed into an infinite sum of the following sinusoid

$$f(t) \sim a_0 + \sum_k a_k \cos(kt) + b_k \sin(kt) \tag{1.6}$$

According to (1.6), signals can be conveniently compressed and denoised. In fact, since the signal frequency is always limited in a certain range in practice, there exists a natural number N satisfying $a_k = b_k = 0$ when $|k| > N$. It shows that if it occurs $|k| > N$ but $(a_k, b_k) \neq (0, 0)$ when the received real signals is decomposed into sinusoids, then the signal is mixed with noise; therefore directly setting $a_k = b_k = 0$ we can achieve the goal of wiping off high-frequency noise. In addition, we know $\lim\limits_{k\to+\infty} a_k = \lim\limits_{k\to+\infty} b_k = 0$ from Riemann-Lebesgue lemma, which indicates that the value of high frequency component of a general real finite energy is becoming smaller when the frequency k becoming larger. Main component of the signal is controlled by a few coefficients (amplitude), so assuming threshold value ε, we can achieve the goal of signal compression in high fidelity when set $a_k = b_k = 0$ as long as $|a_k| < \varepsilon, |b_k| < \varepsilon$.

The two examples above show the application of Fourier series. But there are some problems in the examples when expanding according to Fourier series. For instance, how to determine coefficient b_k in (1.5), how to get the solution in example 1.1 when the length of conductor is not π, how to decompose signal when signal continuance is generally $l, l \leqslant +\infty$. These problems will be solved in the following sections.

1.2 Fourier series preliminary

This section focuses on Fourier series expansion. Firstly we assume the domain of function is $[-\pi, \pi]$, and then discuss general intervals. The following basic conclusions are needed to be established.

Lemma 1.1 (orthogonality of trigonometric base functions) A set consisting of trigonometric base functions

$$\left\{\cdots, \frac{\cos(2x)}{\sqrt{\pi}}, \frac{\cos(x)}{\sqrt{\pi}}, \frac{1}{\sqrt{2\pi}}, \frac{\sin(x)}{\sqrt{\pi}}, \frac{\sin(2x)}{\sqrt{\pi}}, \cdots\right\}$$

are orthogonal on $[-\pi, \pi]$, namely

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nx)\sin(mx)\mathrm{d}x = 0, \quad n, m \in \mathbf{Z}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\mathrm{d}x = \begin{cases} 1, & n = m \geqslant 1 \\ 2, & n = m = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\mathrm{d}x = \begin{cases} 1, & n = m \geqslant 1 \\ 0, & \text{otherwise} \end{cases} \tag{1.7}$$

Proof The proof of the lemma needs the following the product and sum formulae of trigonometric functions

$$\cos x \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$$
$$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$$
$$\sin x \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$$

Thus, when $m \neq n$,

$$\begin{aligned}\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\mathrm{d}x &= \frac{1}{2}\int_{-\pi}^{\pi}[\cos(m+n)x + \cos(m-n)x]\mathrm{d}x \\ &= \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]\Bigg|_{-\pi}^{\pi} = 0\end{aligned}$$

When $m = n \geqslant 1$,

$$\int_{-\pi}^{\pi} \cos^2(nx)\mathrm{d}x = \int_{-\pi}^{\pi} \frac{1}{2}\left(1 + \cos 2nx\right)\mathrm{d}x = \pi$$

Thus the second expression in (1.7) holds. As for the other two expressions, the proof is nearly the same and therefore omitted.

Based on Lemma 1.1, we can prove the Fourier expansion theorem of the following functions.

Theorem 1.1 Suppose the Fourier series expansion of the function $f(x)$ defined on $[-\pi, \pi]$ is

$$f(x) = a_0 + \sum_{k=1}^{+\infty}[a_k \cos(kx) + b_k \sin(kx)] \tag{1.8}$$

Then its Fourier coefficients satisfy

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x \tag{1.9}$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\mathrm{d}x, \quad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\mathrm{d}x, \quad k = 1, 2, 3, \cdots \tag{1.10}$$

Proof For convenience, only a_k is computed.

Multiplying both sides of (1.8) by $\cos(kx)$ and doing integral operation, utilizing Lemma 1.1 for $k = 1, 2, 3, \cdots$, we can get

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos(kx)\mathrm{d}x &= \int_{-\pi}^{\pi}\left(a_0 + \sum_{n=1}^{+\infty} a_n\cos(nx) + b_n\sin(nx)\right)\cos(kx)\mathrm{d}x \\ &= a_k\int_{-\pi}^{\pi}\cos(kx)\cos(kx)\mathrm{d}x \\ &= a_k\pi\end{aligned}$$

Then the first expression in (1.10) holds. The other expressions can be obtained in a similar way.

We need to clarify that, for a 2π- periodic function $G(x)$ defined on $[a, a+2\pi]$, $\int_a^{a+2\pi} G(x)\mathrm{d}x$ $= \int_{-\pi}^{\pi} G(x)\mathrm{d}x$ can be easily proved. Thus, using this expression together with (1.8) and (1.9), it is not difficult to know that the Fourier series expansion of a periodic function defined in any interval of 2π is the same as (1.8).

Now discuss the problem of Fourier series expansion of the function defined in any length l interval. Supposing the function $f(x)$ defined on $[a, b]$, and $l = b - a$, in order to utilize the property of the Fourier series expansion on $[-\pi, \pi]$, performing a linear transformation $x = \frac{l}{2\pi}t + \frac{a+b}{2}$, then when $t \in [-\pi, \pi]$, x is an element in $[a, b]$. Then construct a function $f(x) = f\left(\frac{l}{2\pi}t + \frac{a+b}{2}\right) = F(t)$, so $F(t)$ is a 2π-periodic function defined on $[-\pi, \pi]$. Using Theorem 1.1, $F(t)$ has the following Fourier series expansion.

$$f(x) = F(t) = a_0 + \sum_{k=1}^{+\infty} a_k \cos(kt) + b_k \sin(kt)$$

where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(t)\mathrm{d}t$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} F(t) \cos(kt)\mathrm{d}t, \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} F(t) \sin(kt)\mathrm{d}t, \quad k = 1, 2, 3, \cdots$$

Since $t = \frac{2\pi}{l}x - \frac{a+b}{l}\pi$, we obtain

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(t)\mathrm{d}t = \frac{1}{l} \int_a^b f(x)\mathrm{d}x \tag{1.11}$$

$$\begin{aligned} a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} F(t) \cos(kt)\mathrm{d}t = \frac{2}{l} \int_a^b f(x) \cos\left(\frac{2}{l}x - \frac{a+b}{l}\right) k\pi\mathrm{d}x \\ b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} F(t) \sin(kt)\mathrm{d}t = \frac{2}{l} \int_a^b f(x) \sin\left(\frac{2}{l}x - \frac{a+b}{l}\right) k\pi\mathrm{d}x, \quad k = 1, 2, 3, \cdots \end{aligned} \tag{1.12}$$

and

$$f(x) = a_0 + \sum_{k=1}^{+\infty} \left[a_k \cos\left(\frac{2}{l}x - \frac{a+b}{l}\right) k\pi + b_k \sin\left(\frac{2}{l}x - \frac{a+b}{l}\right) k\pi\right] \tag{1.13}$$

Then we shall study the convergence of Fourier series (1.8) and (1.9), which mainly consists of three types: pointwise convergence, uniform convergence, and mean convergence. First of all, a sufficient condition of pointwise convergence is to be discussed.

Theorem 1.2 If $f(x)$ is a 2π-periodic piecewise continuous function having left and right derivative function at every point, then Fourier series of function $f(x)$ at each point x converge to

$$\frac{f(x+0)+f(x-0)}{2} \tag{1.14}$$

Proof Firstly assume $f(x)$ is continuous and differential at the point x, the partial sum of whose Fourier expansion is

$$S_N(x) = a_0 + \sum_{k=1}^{N} [a_k \cos(kx) + b_k \sin(kx)]$$

Using (1.9) and (1.10), the expression above can be equally written as

$$S_N(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2} + \sum_{k=1}^{N}(\cos(kt)\cos(kx) + \sin(kt)\sin(kx))\right] \mathrm{d}t$$

Using the formula of triangle summation we can get

$$\frac{1}{2} + \sum_{k=1}^{N}[\cos(kt)\cos(kx) + \sin(kt)\sin(kx)] = \frac{1}{2} + \sum_{k=1}^{N} \cos k(t-x)$$
$$= \begin{cases} \dfrac{\sin((N+1/2)(t-x))}{2\sin\dfrac{t-x}{2}}, & t \neq x \\ N+1/2, & t = x \end{cases}.$$

Therefore

$$S_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\left(\frac{\sin(N+1/2)(t-x)}{\sin((t-x)/2)}\right)\mathrm{d}t \equiv \int_{-\pi}^{\pi} f(t)P_N(t-x)\mathrm{d}t \tag{1.15}$$

where

$$P_N(y) = \frac{\sin(N+1/2)y}{2\pi\sin(y/2)}$$

is called Possion kernel. Obviously (let $f(x)=1$) we get

$$\int_{-\pi}^{\pi} P_N(y)\mathrm{d}y = 1 \tag{1.16}$$

Since the period of $f(x)$ and $P_N(x)$is 2π, we have

$$S_N(x) = \int_{-\pi}^{\pi} f(t)P_N(t-x)\mathrm{d}t = \int_{-\pi-x}^{\pi-x} f(y+x)P_N(y)\mathrm{d}y = \int_{-\pi}^{\pi} f(y+x)P_N(y)\mathrm{d}y \tag{1.17}$$

(1.16) and (1.17) imply

$$f(x) - S_N(x) = \int_{-\pi}^{\pi} [f(x) - f(x+y)]P_N(y)\mathrm{d}y$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{f(x)-f(x+y)}{\sin\dfrac{y}{2}}\right)\sin[(N+1/2)y]\mathrm{d}y$$

Denoting $F(y) = \dfrac{f(x)-f(x+y)}{\sin\dfrac{y}{2}}$, because $\lim\limits_{y\to 0} F(y) = -2f'(x)$ shows $F(y)$ is continuous on $[-\pi,\pi]$, by Riemann-Lebesgue lemma we get

$$\int_{-\pi}^{\pi}\left(\frac{f(x)-f(x+y)}{\sin\dfrac{y}{2}}\right)\sin[(N+1/2)y]\mathrm{d}y \to 0 \quad (N\to +\infty)$$

That is $\lim\limits_{N\to\infty} S_N(x) = f(x)$. Here we proved the convergence theorem of Fourier series when a function is continuous and differential.

Next study the convergence when x is a jump discontinuity of function$f(x)$and both the left and right derivative functions exist.

Note that $P_N(y) = \dfrac{\sin(N+1/2)y}{2\pi\sin(y/2)}$ is an even function and using (1.16) deduce out

$$\int_0^{\pi} P_N(y)\mathrm{d}y = \int_{-\pi}^{0} P_N(y)\mathrm{d}y = \frac{1}{2} \tag{1.18}$$

In order to prove the theorem, separately we prove

$$\int_0^{\pi} f(y+x)P_N(y)\mathrm{d}y \to \frac{f(x+0)}{2},\quad \int_{-\pi}^{0} f(y+x)P_N(y)\mathrm{d}y \to \frac{f(x-0)}{2}$$

By (1.18), the expression above equals to

$$\int_0^{\pi}[f(y+x)-f(x+0)]P_N(y)\mathrm{d}y \to 0,\quad \int_{-\pi}^{0}[f(y+x)-f(x-0)]P_N(y)\mathrm{d}y \to 0 \tag{1.19}$$

In fact, construct function

$$F(y) = \frac{f(x+0)-f(x+y)}{\sin\frac{y}{2}}$$

then obtaining $\lim\limits_{y\to 0} F(y) = -2f'(x+0)$. Therefore, using Riemann-Lebesgue lemma, the first expression in (1.19) holds. Similarly, the second expression also holds.

Now discuss the uniform convergence of the partial sum of Fourier series. $S_n(x)$ uniformly converging to $f(x)$ means the convergence and also the converging speed are independent of the point x, that is, for any given positive number $\varepsilon > 0$, there is a natural number N, when $n \geqslant N$, $|S_n(x) - f(x)| < \varepsilon$ holds for every point $x \in [-\pi,\pi]$; while the negation of uniform convergence means there is a positive number $\varepsilon > 0$, for any given natural number N, there exist $n \geqslant N$ and $x_n \in [-\pi,\pi]$, such that $|S_n(x_n) - f(x_n)| \geqslant \varepsilon$. The discussion above states clearly