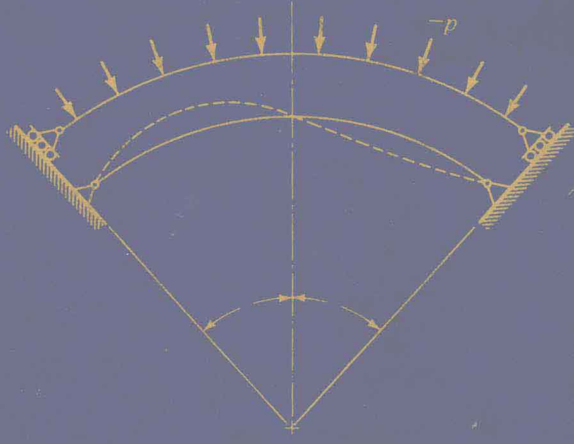
# An Introduction to the ELASTIC STABILITY OF STRUCTURES

GEORGE J. SIMITSES

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An Introduction

to the

# ELASTIC STABILITY OF STRUCTURES

PRENTICE-HALL CIVIL ENGINEERING
AND ENGINEERING MECHANICS SERIES

*N. M. Newmark and W. J. Hall,*
*Editors*

*To my children John, William, and Alexandra,*
*my wife Nena,*
*and my parents John and Vasilike*

# PREFACE

Knowledge of structural stability theory is of paramount importance to the practicing structural engineer. In many instances, buckling is the primary consideration in the design of various structural configurations. Because of this, formal courses in this important branch of mechanics are available to students in Aerospace Engineering, Civil Engineering, Engineering Science and Mechanics, and Mechanical Engineering at many institutions of higher learning. This book is intended to serve as a text in such courses. The emphasis of the book is on the fundamental concepts and on the methodology developed through the years to solve structural stability problems.

The material contained in this text is ideally suited for a one-semester Master's level course, although with judicious addition or deletion of topics, the text may be adopted for both a two-quarter series or a one-quarter course.

The first chapter introduces the basic concepts of elastic stability and the approaches used in solving stability problems, it also discusses the different buckling phenomena that have been observed in nature. In Chapter 2, the basic concepts and methodology are applied to some simple mechanical models with finite degrees of freedom. This is done to help the student understand the fundamentals without getting involved with lengthy and complicated mathematical operations, which is usually the case when dealing with the continuum (infinitely many degrees of freedom). In Chapter 3, a complete treatment of the elastic stability of columns is presented, including effects of elastic restraints. Some simple frame problems are discussed in

Chapter 4. This chapter is of special importance to the Civil Engineering student. Since energy-based methods have been successfully used in structural mechanics, Chapter 5 presents a comprehensive treatment of the energy criterion for stability and contains many energy-related methods. The study of this chapter requires some knowledge of work- and energy-related principles and theorems. These topics are presented in the Appendix for the benefit of the student who never had a formal course in this area. Columns on elastic foundations are discussed in Chapter 6. Chapter 7 presents a comprehensive treatment of the buckling of thin rings and high and low arches. In this chapter, a complete analysis is given for a shallow, pinned, sinusoidal arch on an elastic foundation subject to a sinusoidal transverse loading. This is an interesting model for stability studies because, depending upon the values of the different parameters involved, it exhibits all types of buckling that have been observed in different structural systems: top-of-the-knee buckling, stable bifurcation (Euler-type), and unstable bifurcation. Finally, Chapter 8 contains some remarks about stability of nonconservative elastic systems and dynamic stability. The purpose of this chapter is to motivate the student for further studies by using the references cited.

Once the student has been exposed to the contents of this text, he may, depending upon his interest, proceed with the study of the stability analysis of other structural configurations such as plates, shells, and torsional and lateral buckling of thin-walled open-section beams.

Numerous references are listed at the end of each chapter. These references provide an excellent source for further studies, for better understanding of certain specific concepts, and for detailed information about specific applications.

The author is indebted to the late Professor J. N. Goodier whose two-course series at Stanford University provided the basis for the organization of the material in the present text. The encouragement and valuable suggestions of Professor N. J. Hoff are greatly appreciated. Special thanks are due to Professor M. E. Raville for providing tangible and intangible support, reading the manuscript, and making many corrections. The numerous discussions with Professors S. Atluri, W. W. King, G. M. Rentzepis, C. V. Smith, Jr., and M. Stallybrass are gratefully acknowledged. The proofreading was done by many of my students, but special thanks are due particularly to Dr. V. Ungbhakorn and Mr. J. Giri. Mr. Giri also made most of the drawings. Mrs. Ruth Salley and Mrs. Jackie Van Hook worked with great dedication in typing the manuscript.

G.J.S.

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# 1

# INTRODUCTION AND FUNDAMENTALS

## 1.1 MOTIVATION

Many problems are associated with the design of modern structural systems. Economic factors, availability and properties of materials, interaction between the external loads (aerodynamic) and the response of the structure, dynamic and temperature effects, performance, cost, and ease of maintenance of the system are all problems which are closely associated with the synthesis of these large and complicated structures. Synthesis is the branch of engineering which deals with the design of a system for a given mission. Synthesis requires the most efficient manner of designing a system (i.e., most economical, most reliable, lightest, best, and most easily maintained system), and this leads to *optimization*. An important part of system optimization is structural optimization, which is based on the assumption that certain parameters affecting the system optimization are given (i.e., overall size and shape, performance, nonstructural weight, etc.) It can only be achieved through good theoretical analyses supported by well-planned and well-executed experimental investigations.

Structural analysis is that branch of structural mechanics which associates the behavior of a structure or structural elements with the action of external causes. Two important questions are usually asked in analyzing a structure: (1) What is the response of the structure when subjected to external causes (loads and temperature changes)? In other words, if the external causes are known, can we find the deformation patterns and the internal load distribution? (2) What is the character of the response? Here we are interested in knowing if the equilibrium is stable or if the motion is limited (in the case of dynamic causes). For example, if a load is periodically applied, will the structure oscillate within certain bounds or will it tend to move without bounds?

If the dynamic effects are negligibly small, in which case the loads are said to be applied quasistatically, then the study falls in the domain of structural *statics*. On the other hand, if the dynamic effects are not negligible, we are dealing with structural *dynamics*.

The branch of structural statics that deals with the character of the response is called stability or instability of structures. The interest here lies in the fact that stability criteria are often associated directly with the load-carrying capability of the structure. For example, in some cases instability is not directly associated with the failure of the overall system, i.e., if the skin wrinkles, this does not mean that the entire fuselage or wing will fail. In other cases though, if the portion of the fuselage between two adjacent rings becomes unstable, the entire fuselage will fail catastrophically. Thus, stability of structures or structural elements is an important phase of structural analysis, and consequently it affects structural synthesis and optimization.

## 1.2 STABILITY OR INSTABILITY OF STRUCTURES

There are many ways a structure or a structural element can become unstable, depending on the structural geometry and the load characteristics. The spatial geometry, the material along with its distribution and properties, the character of the connections (riveted joints, welded, etc.), and the supports comprise the structural geometry. By load characteristics we mean spatial distribution of the load, load behavior (whether or not the load is affected by the deformation of the structure, e.g., if a ring is subjected to uniform radial pressure, does the load remain parallel to its initial direction, does it remain normal to the deformed ring, or does it remain directed towards the initial center of curvature?), and/or whether the force system is conservative.

### 1.2-1 Conservative Force Field

A mechanical system is conservative if subjected to conservative forces. If the mechanical system is rigid, there are only external forces; if the system is deformable, the forces may be both external and internal. Regardless of the composition, a system is conservative if all the forces are conservative. A force acting on a mass particle is said to be conservative if the work done by the force in displacing the particle from position 1 to position 2 is independent of the path. In such a case, the force may be derived from a potential. A rigorous mathematical treatment is given below for the interested student.

The work done by a force $\vec{F}$ acting on a mass particle in moving the particle from position $P_0$ (at time $t_0$) to position $P_1$ (at time $t_1$) is given by

$$W = \oint_{C\,\vec{r}_0}^{\vec{r}_1} \vec{F} \cdot d\vec{r} \tag{1}$$

Thus the integral, $W$ (a scalar), depends on the initial position, $\vec{r}_0$, the final position, $\vec{r}_1$, and the path $C$. If a knowledge of the path $C$ is not needed and the work is a function of the initial and final positions only, then

$$W = W(\vec{r}_0, \vec{r}_1, \vec{F}) \tag{2}$$

and the force field is called *conservative*. (See Refs. 1–3.)

**Parenthesis.** If $S$ denotes some surface in the space and $C$ some space curve, then by *Stokes' theorem*

$$\oint_C \vec{U} \cdot d\vec{l} = \iint_S \operatorname{curl} \vec{U} \cdot \vec{n} \, ds \tag{3}$$

where $\vec{n}$ is a unit vector normal to the surface $S$ (see Fig. 1-1).

If $\oint_C \vec{U} \cdot d\vec{l} = 0$, then

$$\iint_S \operatorname{curl} \vec{U} \cdot \vec{n} \, ds = 0 \tag{4}$$

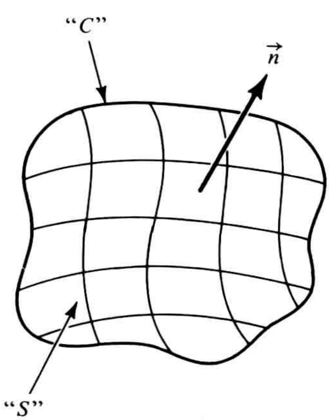


**Figure 1-1.**

for all surfaces $S$ and spanning curves $C$. If this is so, then the curl of $\vec{U}$ (some vector quantity) must be identically equal to zero, or

$$\text{curl } \vec{U} \equiv 0 \tag{5}$$

Next, if we apply this result to a conservative force field where $\vec{U}$ is replaced by $\vec{F}$, then according to the previous result

$$\text{curl } \vec{F} \equiv 0$$

It is well known from *vector analysis* that the curl of the gradient of any scalar function vanishes identically. Therefore, for a conservative field we may write

$$\vec{F} = -\nabla V \tag{6}$$

where:

1. The negative sign is arbitrary,
2. $V$ is some scalar function, and
3. $\nabla$ is the vector operator

$$\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

where $\vec{i}, \vec{j}, \vec{k}$ form an orthogonal unit vector triad along $x, y, z$, respectively. This implies that the force can be derived from a potential.

Note that in this case the work done by the force in a conservative force field is given by

$$W = \oint_{\vec{r}_0}^{\vec{r}_1} \vec{F}\cdot d\vec{r} = -\oint_{\vec{r}_0}^{\vec{r}_1} \nabla V\cdot d\vec{r} = -\oint_{\vec{r}_0}^{\vec{r}_1} (\nabla\cdot d\vec{r})V$$

and since

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k} \quad \text{and} \quad d\vec{r} = (dx)\vec{i} + (dy)\vec{j} + (dz)\vec{k}$$

then

$$(\nabla\cdot d\vec{r})V = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = dV$$

or

$$W = -\int_{V_0}^{V_1} dV = V_0 - V_1 = -\delta(V) \tag{7}$$

where $\delta$ denotes a change in the potential of the conservative force $\vec{F}$ from position $\vec{r}_0$ to position $\vec{r}_1$.

Thus a system is conservative if the work done by the forces in displacing the system from deformation state 1 to deformation state 2 is independent of the path. If this is the case, the force can be derived from a potential.

There are many instances where systems are subjected to loads which cannot be derived from a potential. For instance, consider a column clamped at one end and subjected to an axial load at the other, the direction of which is tangential to the free end at all times (follower force). Such a system is nonconservative and can easily be deduced if we consider two or more possible paths that the load can follow in order to reach a final position. In each case the work done will be different. Systems subject to time-dependent loads are also nonconservative. Nonconservative systems have been given special consideration (Refs. 4 and 5), and the emphasis in this text will be placed on conservative systems (see Ref. 6 for a detailed description of forces and systems).

### 1.2-2 The Concept of Stability

As the external causes are applied quasistatically, the elastic structure deforms and static equilibrium is maintained. If now at any level of the external causes "small" external disturbances are applied and the structure reacts by simply performing oscillations about the deformed equilibrium state, the equilibrium is said to be *stable*. The disturbances can be in the form of deformations or velocities, and by "small" we mean as small as desired. As a result of this latter definition, it would be more appropriate to say that the equilibrium is stable in the small. In addition, when the disturbances are applied, the level of the external causes is kept constant. On the other hand, if the elastic structure either tends to and does remain in the disturbed position or tends to and/or diverges from the deformed equilibrium state, the equlibrium is said to be *unstable*. Some authors prefer to distinguish these two conditions and call the equilibrium *neutral* for the former case and *unstable* for the latter. When either of these two cases occurs, the level of the external causes is called *critical*.

This can best be demonstrated by the system shown in Fig. 1-2. This system consists of a ball of weight $W$ resting at different points on a surface with zero curvature normal to the plane of the figure. Points of zero slope on the

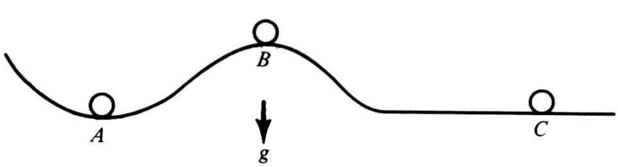


**Figure 1-2.** Character of static equilibrium positions.

surface denote positions of static equilibrium (points $A$, $B$, and $C$). Furthermore, the character of equilibrium at these points is substantially different. At $A$, if the system is disturbed through infinitesimal disturbances (small displacements or small velocities), it will simply oscillate about the static equilibrium position $A$. Such equilibrium position is called *stable* in the small. At point $B$, if the system is disturbed, it will tend to move away from the static equilibrium position $B$. Such an equilibrium position is called *unstable* in the small. Finally, at point $C$, if the system is disturbed, it will tend to remain in the disturbed position. Such an equilibrium position is called *neutrally stable* or *indifferent* in the small. The expression "in the small" is used because the definition depends on the *small* size of the perturbations. If the disturbances are allowed to be of finite magnitude, then it is possible for a system to be unstable in the small but stable in the large (point $B$, Fig. 1-3a) or stable in the small but unstable in the large (point $A$, Fig. 1-3b).

In most structures or structural elements, loss of stability is associated with the tendency of the configuration to pass from one deformation pattern to another. For instance, a long, slender column loaded axially, at the critical condition, passes from the straight configurations (pure compression) to the combined compression and bending state. Similarly, a perfect, complete, thin, spherical shell under external hydrostatic pressure, at the critical condition, passes from a pure membrane state (uniform radial displacement only; shell stretching) to a combined stretching and bending state (nonuniform radial displacements). This characteristic has been recognized for many years and it was first used to solve stability problems of elastic structures. It allows the analyst to reduce the problem to an eigenvalue problem, and many names have been given to this approach: the classical method, the bifurcation method, the equilibrium method, and the static method.

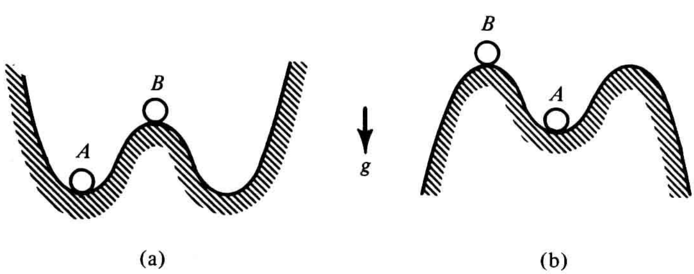


**Figure 1-3.** Character of static equilibrium positions in the large.

### 1.2-3 Critical Loads Versus Buckling Load

At this point nomenclature merits some attention. There is a definite difference in principle between the buckling load observed in a loading pro-