Electric and Magnetic Forces

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PREFACE

All electromagnetic devices that do mechanical work depend for their operation on the forces that arise in the presence of electric and magnetic fields. In simple cases the forces act directly on or between electrical charges—at rest, as in the quadrant electrometer, or in relative motion, as in the familiar domestic television tube. Other devices (e.g. galvanometers) rely on the forces between two or more closed electrical circuits carrying current, and here it is possible to use various alternative formulae for the force between current elements, each of which predicts the force on a complete circuit correctly. However, there are many electromagnetic devices that utilise the forces exerted on polarized or magnetized matter, obvious examples being ordinary relays and electric motors. These *ponderomotive* forces form the subject of the present book: the problem discussed is, in essence, the following. What is the total force (and torque) acting on a polarizable body in an electric field, or a magnetizable body in a magnetic field, and how is this force distributed within and over the body?

Specifying the *location* of ponderomotive forces is much the more difficult part of the problem because the topic is essentially an interdisciplinary one. It originates with the engineer who is interested in electromechanical energy transfer, but it involves the concepts and techniques both of continuum mechanics, which is familiar to an applied mathematician, and of ferromagnetic domain theory, which is the province of the solid state physicist. A successful treatment is one that brings together the relevant ideas from these three separate specialist disciplines: the viewpoint adopted in this book is that it is desirable to accept information from any available source, so that, for example, results derived from microscopic theory will not be rejected—as they sometimes are in continuum mechanics—just *because* they have been deduced microscopically. The interdisciplinary nature of the subject has also, in the past, been the origin of some confusion—real and semantic. For example, mathematicians and engineers have often failed to realise that a magnetic material differs from a continuum in that it is divided up into small regions, or domains, each of which is magnetized to saturation. Whilst this realisation is important it does not lead—as is sometimes supposed—to a facile solution of the problem. It is not feasible to dispense with the familiar framework provided by continuum mechanics; and to consider in detail the domain theory appropriate to a polycrystalline aggregate of single-crystal grains merely results in the wood being obscured by the trees. However, the existence of domains must be kept continually in mind—there is no merit in asserting that the wood does not consist of trees! Physicists, on the other hand, have often been slow to appreciate the practical importance of predictions of ponderomotive forces, since they become aware of the problem only occasionally through, for example, the mechanical failure of a device employing large magnetic fields, such as a synchrotron.

Because of the widespread use of rotating electrical machinery of all types, electrical engineers are, of course, more aware of the problem of 'forces on iron parts'. Some extremely large machines are in use nowadays and manufacturers are understandably reluctant to publicise breakdowns, but there are two types of failure that occur in practice. First, if the rotor becomes displaced from its central position, the total ponderomotive force will be non-zero and may be of sufficient magnitude to 'pull over' the rotor bringing it into contact with the stator, so that mechanical damage and possibly failure follow immediately. This phenomenon is usually initiated by wear or deflection on one or more bearings, and it is important, for example, in large induction machines and high-frequency inductor alternators. Secondly, even when the total ponderomotive force is zero, the force distribution may be such as to lead to failure of parts of the machine. The rotor is, of course, very rigidly constructed but breakages sometimes occur in parts of the stator: the fracture of interpole bolts of traction motors and the fatigue failure of laminations in the stator teeth of large hydrogenerators may be cited as examples. Mechanical resonance is usually important—as it is in the associated problem of noise.

Both electric and magnetic ponderomotive forces are considered but where, for convenience, attention has been concentrated on only one case, the magnetic situation—being, in general, the more difficult—has been chosen. To preserve the symmetry between the treatments of electricity and of magnetism, the pole-strength magnetization $\mathbf{I} = \mathbf{B} - \mu_0\mathbf{H}$ (weber m^{-2}) is used rather than the current-loop magnetization $\mathbf{M} = \mathbf{I}/\mu_0$ (ampere m^{-1}): rationalised m.k.s. units are used throughout. The relative permittivity, $\varepsilon_r = \varepsilon/\varepsilon_o$, and permeability, $\mu_r = \mu/\mu_o$, are not assumed to be constant (or infinite) and are taken, even when not explicitly stated, to be field dependent, i.e. $\varepsilon_r = \varepsilon_r(E)$ and $\mu_r = \mu_r(H)$. In addition, the choice of material for discussion has been influenced by its potential application to polycrystalline bodies of ordinary 'soft' ferromagnetic substances, such as iron: a consideration of materials exhibiting permanent polarization or magnetization is therefore excluded.

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CHAPTER 1

Introduction

1.1 Ponderomotive forces

When a piece of iron is placed in a magnetic field, it is, in general, subject to a force that will move the iron unless it is restrained. Such a force is known as a ponderomotive force. Ponderomotive forces are also exerted on dielectric bodies in electric fields. The material presented in this book refers to static fields only, and it is also assumed that the dielectric or magnetic material is isotropic. A consideration of single crystals and other anisotropic bodies is thus specifically excluded. For convenience, the electric and magnetic cases are treated separately. A problem in which a body that is both dielectric and magnetic is placed in a combined electric and magnetic field must therefore be solved by superposition† of the results for a dielectric body in an electric field and a magnetic body in a magnetic field. When, as in these introductory remarks, it is not necessary to distinguish between the electric and magnetic cases, they will be treated together by referring to the ponderomotive force on a material body in a field.

The interaction between a material body and the field in which it is placed results not only in a tendency for the body to move but also in the body being put into a state of stress. Since all solid materials possess some degree of elasticity this leads to the material being strained—that is, it leads to a deformation of the body. A complete description of the ponderomotive force should therefore include a specification of how the force is distributed throughout the body, although the total ponderomotive force is often the quantity of practical interest. The force distribution depends, of course, on the electric or magnetic field conditions at points within the body.

† Materials that exhibit the unusual magneto-electric effect (Birss 1964) are excluded from consideration.

1.2 Maxwell's field equations

When a material body of arbitrary shape is placed in a known field, the field pattern is modified by the introduction of the body and the specification of the resulting field conditions can itself be quite a formidable problem. The field conditions are completely specified when solutions (satisfying the appropriate boundary conditions) have been obtained to the familiar Maxwell's equations

$$\operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1a}$$

$$\operatorname{div} \mathbf{D} = \rho, \tag{1b}$$

$$\operatorname{curl} \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \tag{1c}$$

$$\operatorname{div} \mathbf{B} = 0. \tag{1d}$$

The solution of these equations is possible only if additional *constitutive* relations are available connecting **D** to **E**, **J** to **E** and **B** to **H**, such as $\mathbf{D} = \varepsilon_r \varepsilon_o \mathbf{E}$, $\mathbf{J} = \sigma \mathbf{E}$, $\mathbf{B} = \mu_r \mu_o \mathbf{H}$ for a linear isotropic material, or some more general relations for a non-linear material. Maxwell's equations (1) constitute a set of coupled first-order partial differential equations connecting the various spatial components of the electric and magnetic vectors. They can sometimes be solved directly but it is often convenient to introduce potential functions and so obtain a smaller number of second-order equations, whilst satisfying some of Maxwell's equations identically.

For static fields, Maxwell's equations reduce to

$$\operatorname{curl} \mathbf{E} = 0, \tag{2a}$$

$$\operatorname{div} \mathbf{D} = \rho, \tag{2b}$$

$$\operatorname{curl} \mathbf{H} = \mathbf{J}, \tag{2c}$$

$$\operatorname{div} \mathbf{B} = 0, \tag{2d}$$

and the vector with the vanishing curl can be written as the (negative) gradient of a scalar potential, Φ, thus

$$\mathbf{E} = -\operatorname{grad}\Phi, \tag{3}$$

whilst the vector with the vanishing divergence can be written as the curl of a vector potential, **A**, thus

$$\mathbf{B} = \operatorname{curl}\mathbf{A}. \tag{4}$$

The definition of **E** and **B** in terms of Φ and **A** ensures that (2a) and (2d) are satisfied identically whilst (2b) and (2c) may be rewritten, in terms of the potentials Φ and **A**. It may be noted that, in contrast to equations (1), equations (2) do not couple together electrical and magnetic vectors: for *time-varying* fields the decoupling process is achieved—as indicated in section 2.3(c)—by exploiting the arbitrariness involved in the definitions of the potential functions. If the potentials Φ and **A** are known as functions of position throughout a region of space, the vectors **E** and **B** may be found from equations (3) and (4) whilst **D** and **H** may be found from the constitutive relations.

1.3 Solutions in free space

In free space, the potential Φ must satisfy the equation obtained by taking the divergence of both sides of (3), namely

$$\nabla^2\Phi = -\frac{\rho}{\varepsilon_0}, \tag{5}$$

which is known as Poisson's equation. If a region is free of charges, the field being produced by charges that can be excluded from the region by closed surfaces drawn around them, then Poisson's equation reduces in the charge-free region to Laplace's equation,

$$\nabla^2\Phi = 0. \tag{6}$$

This is a second-order partial differential equation and its solution for each charge-free region contains two arbitrary functions that can be determined from a knowledge either of the potential at all surfaces or of the potential and the first (spatial) derivatives of the potential at a selected number of surfaces. A solution to equation (6) that satisfies the prescribed boundary conditions gives the value of the potential Φ at every point of the region. An alternative approach is to express Φ as an integral, over a volume V containing all the charges, of a function that involves the charge density ρ. Thus a volume element dV, the position of which is defined by a vector **r**, contains a charge $\rho\, dV$ and its contribution to the potential is, from Coulomb's law,

$$d\Phi = \frac{\rho\, dV}{4\pi\varepsilon_o r}. \tag{7}$$

The total potential due to all the charges is therefore

$$\Phi = \iiint_V \frac{\rho\, dV}{4\pi\varepsilon_o r}, \tag{8}$$

and this formula can be generalised to include a number of bodies and also surface distributions of charge.

In free space, the vector potential **A** must satisfy the equation obtained by taking the curl of both sides of (4), namely

$$\text{curl curl}\,\mathbf{A} \equiv \text{grad div}\,\mathbf{A} - \nabla^2\mathbf{A} = \mu_o\mathbf{J}. \tag{9}$$

Now the divergence of the vector potential **A** is not determined uniquely by equation (4) and div **A** is undefined to the extent of the addition of an arbitrary function of position (see 2.3(c)). For static fields it is customary to make the simplest choice and to set div **A** = 0, since this can be done without altering equation (4). Equation (9) may therefore be written in the form

$$\Delta\mathbf{A} = -\mu_o\mathbf{J}, \tag{10}$$

where the Laplacian operator $\Delta \equiv \nabla^2$ is defined, in curvilinear coordinates by the identity in (9). In Cartesian coordinate systems, however, $\Delta\mathbf{A} \equiv \nabla^2\mathbf{A}$ is a vector the components of which are obtained by operating with ∇^2 on the three components of **A**. The solution of equation (10) subject to arbitrary boundary conditions is usually considerably more complicated than the solution of Laplace's equation (6), because the three components of **A** are not independent but are connected by the equation div **A** = 0. In fact, in the magnetic case it is distinctly advantageous to work with the alternative integral expression obtained from the free-space form of

Biot and Savart's law. For a filamentary circuit l carrying a current i,

$$\mathbf{B} = \frac{\mu_o}{4\pi} i \oint \frac{\mathbf{r} \times d\mathbf{l}}{r^3} \tag{11a}$$

and, for a volume distribution of current,

$$\mathbf{B} = -\frac{\mu_o}{4\pi} \iiint_V \mathbf{J} \times \operatorname{grad} r^{-1} \, dV, \tag{11b}$$

so that the vector potential is given by†

$$\mathbf{A} = \frac{\mu_o}{4\pi} \iiint_V \frac{\boldsymbol{J}}{r} dV. \tag{12}$$

The integrals in equations (8) and (12) are often tedious to evaluate but they represent a method of solution that is suitable for use when a digital computer is available.

1.4 Solutions in the presence of material bodies

If dielectric or magnetic bodies are present, it is only for a few very idealized problems that equations (2) can be solved with any degree of simplicity. Analytic methods, such as those using conformal transformations, do not often lead to solutions in closed form, and solutions in terms of infinite series must converge fairly rapidly to be useful. The practical problems in which the boundary conditions are simple enough to permit an analytic solution (Hauge 1929, Weber 1950, Bewley 1948) are not numerous and are effectively confined to two-dimensional problems. For practical applications experimental and numerical methods of mapping fields have been devised, as well as graphical and semi-graphical procedures involving some calculations.

Perhaps the two best-known approaches are finite-difference methods (Allen 1954, Shaw 1953, Southwell 1940, 1946, 1956) and the electrolytic tank analogue method (Diggle and Harthill 1954). In a

† Equations (12) and (8) are discussed further below.

finite-difference method, the differential equation for a potential is replaced by many finite-difference equations which take the form of linear equations connecting the potential at one of a finite number of regularly spaced points, or nodes, with the potentials at other nodes close to it. Trial values of the potentials at the nodes are then successively improved by considering the effects at adjacent points of a change in the potential at a particular node. Non-linearity in a constitutive relation can be accommodated by a process of successive approximation or, more usually, by replacing the constitutive relation by two or more linear relations. In the case of the magnetization of iron bodies, for example, saturation is often important, and this can be allowed for in an approximate fashion by replacing the magnetization curve (B versus H) with a similarly shaped curve constructed by connecting two or more straight line segments of differing slopes. In the electrolytic tank analogue method, advantage is taken of the analogy between the electric field produced by a point charge and the pattern of flow of current in a conducting electrolytic solution when current is fed into it at a corresponding point. Since the current is subject to the continuity equation

$$\operatorname{div} \mathbf{J} + \frac{\partial \rho}{\partial t} = 0\,, \tag{13}$$

where $\partial \rho / \partial t$ is zero except at points at which current is being fed into (or removed from) the electrolyte, there is a direct analogy with equation (2b). The electrolytic tank method can thus readily be used to predict the electric field due to a collection of point charges or a distribution of charges. It can also be used to determine the magnetic field due to a collection of line currents, provided that the problem is sensibly two-dimensional, for it is then possible to interchange the roles of lines of force and lines of equipotential in the analogue. An advantage of the method is that in two-dimensional problems it is possible to allow for non-linearity in a constitutive relation by using a tank in which the depth of electrolyte varies from place to place.

The specification of the field conditions that result when a material body of arbitrary shape is placed in a known field can be, as mentioned above, quite a formidable problem, and it will not be pursued further here. There is an extensive coverage of this problem in the literature† and it will be assumed in what follows that the field conditions can always be determined uniquely at points outside and within the body. The problem of interest therefore is to specify the distribution of ponderomotive force in terms of these known field conditions.

1.5 Vector fields and their singularities

Having taken for granted that a solution—exact or approximate—can always be found to the static form (2) of Maxwell's equations, it is desirable to consider what is involved in specifying field conditions within a material body by the four vectors **E**, **D**, **H** and **B**, and indeed why four are necessary rather than just one electrical and one magnetic vector. In free space there is no need to distinguish between the two electric vectors **E** and **D** or between the two magnetic vectors **H** and **B**. Moreover, a simple consideration of static electric and magnetic fields reveals a difference between them in that they exhibit different sorts of singularities. A graphical display, or field plot, of the lines of force can reveal (in a favourably orientated cross-section) two sorts of singularities typified by the diagrams shown in Fig. 1.1. These are both singularities in the field, and

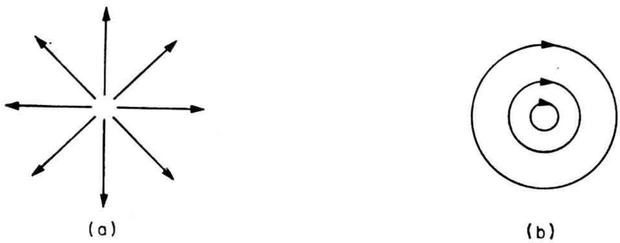

Fig. 1.1

appear to act in some way as *origins* of the field. A singularity of type (a) will be referred to as a source and a singularity of type (b) as a vortex. It is a matter of experiment that sources are typical of

† Interested readers may care to consult, for example, the book by K. J. Binns and P. J. Lawrenson, *Analysis and Computation of Electric and Magnetic Field Problems*, Pergamon Press, 1963.

electric fields and are associated with the presence of charge whereas vortices are typical of magnetic fields (produced electromagnetically) and are associated with current.† Of course, both an indefinitely small single charge and an indefinitely thin wire carrying current are idealizations, but they are idealizations that are approximately realized in practice. It is possible, for example, to experiment not only with small charged material bodies but also with very small fundamental particles (electrons, protons, etc.) that carry discrete quantities of charge, whilst the approximation of neglecting the thickness of a conductor carrying current is a common one in electrical engineering. It may be mentioned in passing that the laws relating fields to charges and line currents, once obtained, can be readily extended to deal with surface and volume distributions of charge and current.

The *strength* of a source or a vortex is a measure of how much of the field originates at the singularity. If Fig. 1.1(a) represents lines of force in some vector field $\mathbf{F}$ (for example, $\mathbf{F} \equiv \mathbf{E}$), the strength of the source depends on how many lines radiate outwards from the singularity. Remembering that Fig. 1.1(a) is a two-dimensional representation of a source in three dimensions, it may be seen that a geometrically reasonable definition of the strength of a source is given by

$$s' = \oiint_S \mathbf{F} \, . \, d\mathbf{S}, \tag{14}$$

where S is a closed surface surrounding the singularity. However, many sources may be present simultaneously, so it is customary to replace s' by a source density $s = s'/V$, where V is the volume enclosed by the surface S, and to proceed to the limit when the volume contracts whilst still containing the singularity. The source density s is therefore defined by the equation

$$s = \operatorname*{Lim}_{V \to 0} \left[\frac{1}{V} \oiint_S \mathbf{F} \, . \, d\mathbf{S} \right] \equiv \operatorname{div} \mathbf{F}. \tag{15}$$

† See 1.6 for a discussion of the existence of magnetic monopoles.