# Perturbations
## Theory and Methods

James A. Murdock

C · L · A · S · S · I · C · S

In Applied Mathematics

siam

27

# Perturbations
## Theory and Methods

James A. Murdock
Iowa State University
Ames, Iowa





siam
Society for Industrial and Applied Mathematics
Philadelphia

This SIAM edition is an unabridged, corrected republication of the work first published by John Wiley & Sons, Inc., New York, 1991.

10 9 8 7 6 5 4 3 2 1



**Library of Congress Cataloging-in-Publication Data**

Murdock, James A.

Perturbations : theory and methods / James A. Murdock.

p. cm. -- (Classics in applied mathematics ; 27)

Includes bibliographical references and index.

ISBN 0-89871-443-5

1. Perturbation (Mathematics) I. Title II. Series.

QA871.M87 1999

515'.35--dc21 99-36132

CIP

# Perturbations

SIAM's Classics in Applied Mathematics series consists of books that were previously allowed to go out of print. These books are republished by SIAM as a professional service because they continue to be important resources for mathematical scientists.



**Classics in Applied Mathematics**

C. C. Lin and L. A. Segel, *Mathematics Applied to Deterministic Problems in the Natural Sciences*

Johan G. F. Belinfante and Bernard Kolman, *A Survey of Lie Groups and Lie Algebras with Applications and Computational Methods*

James M. Ortega, *Numerical Analysis: A Second Course*

Anthony V. Fiacco and Garth P. McCormick, *Nonlinear Programming: Sequential Unconstrained Minimization Techniques*

F. H. Clarke, *Optimization and Nonsmooth Analysis*

George F. Carrier and Carl E. Pearson, *Ordinary Differential Equations*

Leo Breiman, *Probability*

R. Bellman and G. M. Wing, *An Introduction to Invariant Imbedding*

Abraham Berman and Robert J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*

Olvi L. Mangasarian, *Nonlinear Programming*

*Carl Friedrich Gauss, *Theory of the Combination of Observations Least Subject to Errors: Part One, Part Two, Supplement.* Translated by G. W. Stewart

Richard Bellman, *Introduction to Matrix Analysis*

U. M. Ascher, R. M. M. Mattheij, and R. D. Russell, *Numerical Solution of Boundary Value Problems for Ordinary Differential Equations*

K. E. Brenan, S. L. Campbell, and L. R. Petzold, *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations*

Charles L. Lawson and Richard J. Hanson, *Solving Least Squares Problems*

J. E. Dennis, Jr. and Robert B. Schnabel, *Numerical Methods for Unconstrained Optimization and Nonlinear Equations*

Richard E. Barlow and Frank Proschan, *Mathematical Theory of Reliability*

*First time in print.

**Classics in Applied Mathematics (continued)**

Cornelius Lanczos, *Linear Differential Operators*

Richard Bellman, *Introduction to Matrix Analysis, Second Edition*

Beresford N. Parlett, *The Symmetric Eigenvalue Problem*

Richard Haberman, *Mathematical Models: Mechanical Vibrations, Population Dynamics, and Traffic Flow*

Peter W. M. John, *Statistical Design and Analysis of Experiments*

Tamer Başar and Geert Jan Olsder, *Dynamic Noncooperative Game Theory, Second Edition*

Emanuel Parzen, *Stochastic Processes*

Petar Kokotović, Hassan K. Khalil, and John O'Reilly, *Singular Perturbation Methods in Control: Analysis and Design*

Jean Dickinson Gibbons, Ingram Olkin, and Milton Sobel, *Selecting and Ordering Populations: A New Statistical Methodology*

James A. Murdock, *Perturbations: Theory and Methods*

To William B. Inhelder,
My High School Mathematics Teacher

# PREFACE TO THE CLASSICS EDITION

On the occasion of the reprinting of my book by SIAM in its Classics in Applied Mathematics series, I have been asked to say a few words about why this book is still relevant and how it relates to the current situation in perturbation theory. This is actually quite easy to do, because as far as the material covered in this book is concerned, nothing has changed. Of course, there have been developments in the field. But this book is an introduction, and the recent developments have occurred at a level beyond what is presented here. In addition to these developments, several new introductions to perturbation theory have been published. But among introductions, mine is still unique in its attempt to present perturbation theory as a natural part of a larger whole, the mathematical theory of differential equations, and to dwell at some length on the meaning of the results and their connections with other ways of studying the same problems. It is still the fashion to treat perturbation theory as a bag of heuristic tricks with no foundation, leading to approximate solutions that are justified only by comparing them with numerical solutions or experimental data. Books of this kind still make the standard error of confusing asymptotic validity with asymptotic ordering. If my book has one central message, it is to clarify this particular issue for the beginning student (and along with it, the related issue of the meaning of various kinds of uniformity).

The reception that my book has received is probably what I should have expected: it has been warmly praised by the leading experts in the field and by the occasional bright student who has found clarity here for the first time, and it has been almost entirely ignored as a book for classroom use. At Iowa State University, where the book and its associated course were developed, perturbation theory is taught as a graduate course in the mathematics department, but elsewhere the subject is mostly taught by engineers. In hindsight, it was probably foolish to hope that many of them would adopt this book. But now that the original printing has sold out, there seems to be a demand for copies from those who do teach or use perturbation theory in a mathematical context. Perhaps publication by SIAM will cause this book to reach a more receptive audience.

The question that I am asked more frequently than any other is why this book does not have more about boundary value problems, especially boundary layer (as opposed to initial layer) problems. There are three parts to the answer: one is personal preference and knowledge, another is deliberate decision, and the third is to say, "Look again; there is more there about boundary values than you might think at first sight." The personal preference part is that I think mostly about dynamical systems, and most dynamical systems problems are initial value problems. The deliberate decision part is connected with the overall scheme of the book, which is to treat both the heuristics and the error estimation for each method. Error estimation for initial value problems is almost always centered around one technique, Gronwall's inequality, whereas error estimation for boundary value problems requires a wide variety of techniques such as Green's functions and various comparison theorems. To present all of these techniques, for first year graduate students who cannot be presumed to have had exposure to them, would have made the book an "introduction to applied mathematics" rather than to perturbation theory. By focusing on initial value problems I was able to present all of the standard perturbation methods, and their rigorous justification, without long digressions on estimation methods other than Gronwall's inequality. The book Singular Perturbation Theory by Donald R. Smith (Cambridge University Press, 1985), mentioned in the preface to the first edition of this book, is a good source for a variety of error estimation techniques that go beyond those developed here. The "look again" part of the answer is to point out that boundary value problems are often treated in this book as a second application of ideas already developed for initial value problems, and that therefore the boundary value problem material does not take up as much space as in most books because it is not necessary to repeat in detail things that have already been covered. This economy of space is another consequence of treating the ideas in relation to one another rather than as disconnected tricks. Examples include the linkage of boundary value problems to initial value problems by the shooting method in section 2.5, and the heavy dependence of the short boundary layer chapter (chapter 8) on the longer initial layer chapter (chapter 7). For instance, there is no need to repeat the extensive discussion of the Van Dyke rules and overlap domains (sections 7.2 and 7.3) in chapter 8.

This edition is a reprint of the original, with no changes other than correction of typographical errors and a few more serious errors. One error that has not been corrected, because it is minor and would require redrawing a complicated figure, is in Figure 4.3.1: the curves intersecting at the unstable rest point do not have a vertical tangent but, rather, cross with two distinct tangents.

*James A. Murdock*

# PREFACE

The beginning student of differential equations quickly exhausts the few types which can be solved "in closed form," that is, using elementary functions: the first order exact, linear, and homogeneous equations; the higher order linear equations with constant coefficients; the partial differential equations that are reducible to these by separation of variables. After this, there are several directions: advanced theory, approximation of solutions on the digital computer by numerical analysis, approximation of solutions by means of formulas. The last is the primary subject of this book. The advantage of having an approximate formula for the solution of an equation, as opposed to having a computer program that generates numbers, is that it is more easily possible to see the role of the different variables and parameters in the solution: to recognize, for instance, the effect of a friction parameter on the frequency of an oscillation. There are, of course, advantages to numerical methods as well, one of the greatest being that they apply over a wider range of the parameters. Perturbation methods, as the name implies, are useful only when the equation to be solved is close to ("is a perturbation of") a solvable equation.

Perturbation theory has the reputation of being a bag of tricks giving formulas that often work but are seldom justifiable. It is viewed this way by many who use it successfully, as well as by those who refuse to use it because of its alleged lack of mathematical rigor. But one only needs to remember that the definition of asymptotic series was due to Henri Poincaré to realize that the separation of perturbation methods from other approaches (such as the geometrical analysis of solution orbits), although common in the classroom today, is a betrayal of the true nature of the subject. Just as one should never feed a differential equation into a computer without having a feeling for the type of solutions to be expected and what numerical difficulties they threaten, one should never (or almost never) apply a perturbation method without at the same time considering the existence and uniqueness of solutions, the solution geometry, possible bifurcations, and other factors that might affect the accuracy and meaning of the solutions.

On the other hand, to present perturbation theory in such a holistic context poses a difficulty. Because of the demand for courses in perturbation theory from the extremely applied end of the mathematical spectrum, and because of the ease of teaching a course that is almost entirely formal in content (a "cookbook" course), it has become popular to teach perturbation theory at a beginning graduate level to students who have not yet mastered, or even been exposed to, the methods of proof needed to establish the existence of the solutions to which approximations are being found. Some of the students in such a course, particularly those coming from outside the mathematics department, may never study these proof techniques. Does this mean that they should be denied an awareness of the extent to which mathematical theory interacts with practical computational methods, especially when it is this very theory that makes it possible to appreciate the conditions under which the methods may fail? On the contrary, it would seem that one of the most important mathematical skills to be acquired by a user of mathematics is precisely the ability to read a mathematical theorem and glean its significance for an application, without necessarily studying its proof. And for the prospective mathematician, who already has some appreciation of the need for proof, an exposure to problem solving based on theorems whose proof will be studied later can only help the student to feel "at home" when the proofs are finally encountered.

Therefore it is the premise of this book that even at a beginning level, perturbation theory should be presented not as an isolated collection of cookbook techniques but as a part of the mainstream of mathematics. This necessarily means that theorems covered in other mathematics courses will be encountered. Our approach is to make these theorems available, whether or not the student has seen them elsewhere, by a clear statement and intuitive discussion in an appendix. The mathematically sophisticated reader will of course not need these appendices, although even such a reader may find bits of intuitive insight fall into place in reading them; I certainly did, as I wrote them. For others, they will serve as examples of how to extract the meaning from a mathematical statement, or as a foretaste of things to be studied later.

Throughout this book, my effort has been to say the important things that no one ever seems to say, the simple insights that finally dawned on me after months (sometimes years) of working with a method and reading its standard literature; the things that made me think: "Why didn't they tell me that in the first place?" If my experience is at all typical, there must be many students who will welcome the discussions in this book, although others may find the book too "chatty" and wish for more formulas and fewer words. Some may feel that there are not enough solved computational examples. My response to this is one of personal temperament. I do not enjoy computation for its own sake, and whenever I attempt a problem, I end up spending two weeks searching into its meaning, its theoretical significance, possible variations, places where it might stretch the limits of existing theory, and so forth. Therefore I do not have a large stock of problems from which to draw for this book; almost every problem I have ever solved, if it had anything worthwhile to contribute, is included here. I can make no claim to the kind of computational virtuosity exhibited in, for instance, the books by Ali Nayfeh. Some of my examples are taken from his books, frequently with additional development. (For instance, the solvable triple-deck boundary value problem taken from Nayfeh and given here in Example 8.1.1 becomes unsolvable when changed to an initial value problem, Example 7.6.2.) This book is in many ways complementary to Nayfeh's books, and certainly does not replace them.

I have made no effort to distinguish between new and old results in the main text of the book. There are many elementary ideas here that I have never seen in print, which I had to develop in the same way as I would a research result, and yet I cannot claim them as my own, since many people must have known them without writing them out explicitly. An example is the discussion of what I call the "trade-off" property of Lindstedt series: A solution that is accurate to order $\mathcal{O}(\varepsilon^k)$ on a time interval of length $\mathcal{O}(1/\varepsilon)$ is also accurate to order $\mathcal{O}(\varepsilon^{k-j})$ on an interval of length $\mathcal{O}(1/\varepsilon^{1+j})$ for $0 < j < k$. This can be proved easily for the Lindstedt method, and yet fails for multiple scale methods in general. The literature is full of vague claims to this effect for both methods. I have not seen the proof for the Lindstedt case before, and yet it is too elementary to claim as new. On the other hand, I have recently given an example which I think makes clear for the first time that there are fundamental limitations on the attempt to extend the multiple scale method to longer intervals of time. This example (in a simplified form) is outlined in the text, without indicating it as mine. Each chapter does end with an annotated "Notes and References" section, which is intended to guide the student to related reading and occasionally to give historical information; the original attribution of results that are not in general circulation is given here when it is known to me. These notes are in no way intended to be exhaustive or up to the minute.

The book is divided into three parts, covering regular perturbation theory, oscillatory phenomena (with the Lindstedt, multiple scale, and averaging methods), and transition layer phenomena (initial layers, boundary layers, turning points, and such). The first two of these parts have been in preparation for a number of years and contain most of the original contributions; the second part, especially, is closely related to my own research and goes far enough to provide initial access to my papers. (But no farther; I have resisted the temptation to unbalance this book too greatly by presenting actual research here.) The third part contains new expositions of classical ideas but does not reach as close to current research as does the second. (The one relatively recent topic mentioned in this part, without much development, is canards.) Largely this is because of my own lack of extensive knowledge in transition layer phenomena. I have tried to learn, and since as a thinker I tend strongly toward the "foundational," I have worked hard at understanding the basics. The result is that students struggling with "matching" for the first time seem to find the approach taken here to be much less mysterious than the usual exposition; that is, if they are interested in more than computational facility.

Most of the notations used are self-explanatory. Boldface is used for vectors, although in advanced mathematics this is usually dispensed with, and students need not feel obligated to underline or otherwise designate vectors in their own work. All vectors are to be treated as column vectors, unless otherwise indicated, whenever matrix multiplication is involved, but we still write them as $\mathbf{x} = (x_1, \ldots, x_N)$ most of the time without indicating "transpose" or "col." A boldfaced function with a boldfaced subscript indicates the matrix of partial derivatives of the components of the function with respect to the components of the subscript; in each row of such a matrix, the component of the function is fixed and the component of the subscript varies. As a general rule, $N$ and $M$ are used for dimensions of vectors, $n$ for the general term of a finite or infinite series, and $k$ for the last term of a finite series. Other indices are used as needed; it should be clear when $i$ is an index and when it is $\sqrt{-1}$.

There are a few textbooks that serve as general references for additional reading on most topics in this book. These will not be repeated in each "Notes and References" section, except to point out something particularly good for an individual topic. Therefore they will be listed here, with brief comments. The works of Ali Nayfeh are excellent references for the computational aspects of all parts of perturbation theory, but are not reliable in regard to the theory. The most useful are

Ali Nayfeh, *Introduction to Perturbation Techniques*, Wiley, New York, 1981

and

Ali Nayfeh, *Perturbation Methods*, Wiley, New York, 1973.

The former is an introduction to the basic techniques with many worked examples and is in most cases the best book for the student to consult when looking for additional exercises or for an alternative explanation of a topic that is found to be difficult. The latter is an encyclopedic reference covering almost all methods and giving many references. The principal "error" to watch out for in these books is in the interpretation of the big-oh symbol. An equation will be given containing a perturbation parameter $\varepsilon$ and a constant control parameter $\lambda$, and it will be stated that when $\lambda$ is $\mathcal{O}(1)$, one method should be used, but that this method "breaks down" when $\lambda$ is $\mathcal{O}(\varepsilon)$, and then another method (perhaps a rescaling) should be used.

Of course, it is impossible for a constant to be $\mathcal{O}(\varepsilon)$, no matter how small it may be, so the statement is entirely meaningless. The correct statement is that the first method is uniformly valid for $\lambda$ in any compact set (closed bounded interval) not containing zero, whereas the second is uniformly valid for $\lambda$ in a shrinking $\varepsilon$-dependent interval whose length is $\mathcal{O}(\varepsilon)$. Once these matters are clearly understood—and they are explained at length in Section 1.7 below—it is easy to profit from the books by Nayfeh without becoming confused.

The book

J. Kevorkian and J. D. Cole, *Perturbation Methods in Applied Mathematics*, Springer-Verlag, New York, 1981

covers many of the topics in this book, emphasizing the multiple scale and matching methods. The range of applications given is much greater than in this book and includes many difficult examples with partial differential equations. Most of the work is formal, without error estimates, but it is quite carefully done. In a different spirit is

Donald R. Smith, *Singular Perturbation Theory*, Cambridge University Press, Cambridge, 1985.

This book emphasizes what we call direct error estimation and presents many up-to-date theorems about the degree of accuracy of multiple scale and boundary layer correction methods. (This book prefers the correction method to the essentially equivalent matching method for transition layer problems.) The last three books are good starting places for a student wishing to go beyond the present book, but they leave unsaid many of the first principles that are explained here. A reference that includes a variety of perturbation methods explained from a practical point of view using a variety of interesting examples is

Carl M. Bender and Steven A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers*, McGraw-Hill, New York, 1978.

An extensive discussion of perturbation methods in partial differential equations (barely touched on in the present text) is given in Chapter 9 (all 214 pages of it!) of

Erich Zauderer, *Partial Differential Equations of Applied Mathematics*, Wiley, New York, 1983.

## ACKNOWLEDGMENTS

This book would not have been possible without the teachers that encouraged and supported me over the years, and the students who used preliminary drafts of portions of the book in their classes. Among the teachers

were William Inhelder, who knew his epsilons and deltas and believed that real mathematics belonged in high school, where it has now been replaced by mass-produced advanced placement calculus; Wendell Fleming, who thought undergraduates deserved differential forms and Lebesgue integrals; and Jürgen Moser, who introduced me to perturbation theory and criticized some of my early attempts at mathematical writing. Among the students, Chao-Pao Ho did the largest amount of proofreading.

The majority of the graphs (35) were done by Kurt Whitmore, an undergraduate at Iowa State University, using Mathematica and other software on a Macintosh SE/30 with a laser printer; a few were done in the same way by Tony Walker (15), Tom Bullers (9), and Jonathan Schultz (1), also undergraduates. Only Fig. 4.7.4 was drawn by hand.

JAMES A. MURDOCK

# CONTENTS