Numerical Analysis 2000, Volume 5

Quadrature and Orthogonal Polynomials

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Preface

Orthogonal polynomials play a prominent role in pure, applied, and computational mathematics, as well as in the applied sciences. It is the aim of the present volume in the series "Numerical Analysis in the 20th Century" to review, and sometimes extend, some of the many known results and properties of orthogonal polynomials and related quadrature rules. In addition, this volume discusses techniques available for the analysis of orthogonal polynomials and associated quadrature rules. Indeed, the design and computation of numerical integration methods is an important area in numerical analysis, and orthogonal polynomials play a fundamental role in the analysis of many integration methods.

The 20th century has witnessed a rapid development of orthogonal polynomials and related quadrature rules, and we therefore cannot even attempt to review all significant developments within this volume. We primarily have sought to emphasize results and techniques that have been of significance in computational or applied mathematics, or which we believe may lead to significant progress in these areas in the near future. Unfortunately, we cannot claim completeness even within this limited scope. Nevertheless, we hope that the readers of this volume will find the papers of interest and many references to related work of help.

We outline the contributions in the present volume. Properties of orthogonal polynomials are the focus of the papers by Marcellán and Álvarez-Nodarse and by Freund. The former contribution discusses "Favard's theorem", i.e., the question under which conditions the recurrence coefficients of a family of polynomials determine a measure with respect to which the polynomials in this family are orthogonal. Polynomials that satisfy a three-term recurrence relation as well as Szegő polynomials are considered. The measure is allowed to be signed, i.e., the moment matrix is allowed to be indefinite. Freund discusses matrix-valued polynomials that are orthogonal with respect to a measure that defines a bilinear form. This contribution focuses on breakdowns of the recurrence relations and discusses techniques for overcoming this difficulty. Matrix-valued orthogonal polynomials form the basis for algorithms for reduced-order modeling. Freund's contribution to this volume provides references to such algorithms and their application to circuit simulation.

The contribution by Peherstorfer and Steinbauer analyzes inverse images of polynomial mappings in the complex plane and their relevance to extremal properties of polynomials orthogonal with respect to measures supported on a variety of sets, such as several intervals, lemniscates, or equipotential lines. Applications include fractal theory and Julia sets.

Orthogonality with respect to Sobolev inner products has attracted the interest of many researchers during the last decade. The paper by Martinez discusses some of the recent developments

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in this area. The contribution by López Lagomasino, Pijeira, and Perez Izquierdo deals with orthogonal polynomials associated with measures supported on compact subsets of the complex plane. The location and asymptotic distribution of the zeros of the orthogonal polynomials, as well as the nth-root asymptotic behavior of these polynomials is analyzed, using methods of potential theory.

Investigations based on spectral theory for symmetric operators can provide insight into the analytic properties of both orthogonal polynomials and the associated Padé approximants. The contribution by Beckermann surveys these results.

Van Assche and Coussement study multiple orthogonal polynomials. These polynomials arise in simultaneous rational approximation; in particular, they form the foundation for simultaneous Hermite–Padé approximation of a system of several functions. The paper compares multiple orthogonal polynomials with the classical families of orthogonal polynomials, such as Hermite, Laguerre, Jacobi, and Bessel polynomials, using characterization theorems.

Bultheel, González-Vera, Hendriksen, and Njåstad consider orthogonal rational functions with prescribed poles, and discuss quadrature rules for their exact integration. These quadrature rules may be viewed as extensions of quadrature rules for Szegő polynomials. The latter rules are exact for rational functions with poles at the origin and at infinity.

Many of the papers of this volume are concerned with quadrature or cubature rules related to orthogonal polynomials. The analysis of multivariable orthogonal polynomials forms the foundation of many cubature formulas. The contribution by Cools, Mysovskikh, and Schmid discusses the connection between cubature formulas and orthogonal polynomials. The paper reviews the development initiated by Radon's seminal contribution from 1948 and discusses open questions. The work by Xu deals with multivariate orthogonal polynomials and cubature formulas for several regions in $\mathbb{R}^d$. Xu shows that orthogonal structures and cubature formulas for these regions are closely related.

The paper by Milovanović deals with the properties of quadrature rules with multiple nodes. These rules generalize the Gauss–Turán rules. Moment-preserving approximation by defective splines is considered as an application.

Computational issues related to Gauss quadrature rules are the topic of the contributions by Ehrich and Laurie. The latter paper discusses numerical methods for the computation of the nodes and weights of Gauss-type quadrature rules, when moments, modified moments, or the recursion coefficients of the orthogonal polynomials associated with a nonnegative measure are known. Ehrich is concerned with how to estimate the error of quadrature rules of Gauss type. This question is important, e.g., for the design of adaptive quadrature routines based on rules of Gauss type.

The contribution by Mori and Sugihara reviews the double exponential transformation in numerical integration and in a variety of Sinc methods. This transformation enables efficient evaluation of the integrals of analytic functions with endpoint singularities.

Many algorithms for the solution of large-scale problems in science and engineering are based on orthogonal polynomials and Gauss-type quadrature rules. Calvetti, Morigi, Reichel, and Sgallari describe an application of Gauss quadrature to the computation of bounds or estimates of the Euclidean norm of the error in iterates (approximate solutions) generated by an iterative method for the solution of large linear systems of equations with a symmetric matrix. The matrix may be positive definite or indefinite.

The computation of zeros of polynomials is a classical problem in numerical analysis. The contribution by Ammar, Calvetti, Gragg, and Reichel describes algorithms based on Szegő polynomials.

In particular, knowledge of the location of zeros of Szegő polynomials is important for the analysis and implementation of filters for time series.

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Polynomial zerofinders based on Szegő polynomials

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Abstract

The computation of zeros of polynomials is a classical computational problem. This paper presents two new zerofinders that are based on the observation that, after a suitable change of variable, any polynomial can be considered a member of a family of Szegő polynomials. Numerical experiments indicate that these methods generally give higher accuracy than computing the eigenvalues of the companion matrix associated with the polynomial. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Szegő–Hessenberg matrix; Companion matrix; Eigenvalue problem; Continuation method; Parallel computation

1. Introduction

The computation of the zeros of a polynomial

$$\psi_n(z) = z^n + \alpha_{n-1}z^{n-1} + \cdots + \alpha_1 z + \alpha_0, \quad \alpha_j \in \mathbb{C}, \tag{1}$$

is a fundamental problem in scientific computation that arises in many diverse applications. The conditioning of this problem has been investigated by Gautschi [8,9]. Several classical methods for determining zeros of polynomials are described by Henrici [17, Chapter 6] and Stoer and Bulirsch [26, Chapter 5]. A recent extensive bibliography of zerofinders is provided by McNamee [21].

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Among the most popular numerical methods for computing zeros of polynomials are the Jenkins–Traub algorithm [18], and the computation of the zeros as eigenvalues of the companion matrix

$$C_n = \begin{bmatrix} 0 & & & \cdots & 0 & -\alpha_0 \\ 1 & 0 & & \cdots & 0 & -\alpha_1 \\ & 1 & 0 & \cdots & 0 & -\alpha_2 \\ & & & & \vdots & \vdots \\ & & \ddots & \ddots & & \\ & & & 1 & 0 & -\alpha_{n-2} \\ 0 & & & & 1 & -\alpha_{n-1} \end{bmatrix} \in \mathbb{C}^{n\times n} \tag{2}$$

associated with the polynomial (1) by the QR algorithm after balancing; see Edelman and Murakami [7] and Moler [22]. Recently, Goedecker [10] compared these methods and found the latter approach to be competitive with several available implementations of the Jenkins–Traub algorithm with regard to both accuracy and execution time for polynomials of small to moderate degree.

This paper describes two new methods for computing zeros of polynomials. The methods are based on the observation that, after a change of variable, any polynomial can be considered a member of a family of Szegő polynomials. The new zerofinders use the recursion relation for the Szegő polynomials, which are defined as follows. Let ω be a nondecreasing distribution function with infinitely many points of increase on the unit circle in the complex plane and define the inner product

$$(f,g):=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(z)\overline{g(z)}\,\mathrm{d}\omega(t), \quad z:=\exp(\mathrm{i}t), \quad \mathrm{i}:=\sqrt{-1}, \tag{3}$$

for polynomials f and g, where the bar denotes complex conjugation. We assume for notational convenience that $\mathrm{d}\omega(t)$ is scaled so that $(1,1)=1$. Introduce orthonormal polynomials with respect to this inner product, $\phi_0,\phi_1,\phi_2,\ldots$, where ϕ_j is of degree j with positive leading coefficient. These polynomials are known as Szegő polynomials and many of their properties are discussed by Grenander and Szegő [16]. In particular, they satisfy the recursion relation

$$\begin{aligned} &\phi_0(z)=\phi_0^*(z)=1, \\ &\sigma_{j+1}\phi_{j+1}(z)=z\phi_j(z)+\gamma_{j+1}\phi_j^*(z), \quad j=0,1,2.,\ldots,n-1, \\ &\sigma_{j+1}\phi_{j+1}^*(z)=\bar{\gamma}_{j+1}z\phi_j(z)+\phi_j^*(z), \end{aligned} \tag{4}$$

where the recursion coefficients γ_{j+1} and the auxiliary coefficients σ_{j+1} are defined by

$$\begin{aligned} &\gamma_{j+1}=-\frac{(z\phi_j,1)}{\delta_j}, \\ &\sigma_{j+1}=\sigma_j(1-|\gamma_{j+1}|^2), \quad j=0,1,2,\ldots, \\ &\delta_{j+1}=\delta_j\sigma_{j+1}, \quad \delta_0=\sigma_0=1. \end{aligned} \tag{5}$$

It follows from (4) that the auxiliary polynomials ϕ_j^* satisfy

$$\phi_j^*(z) := z^j \bar{\phi}_j(1/z). \tag{6}$$

The zeros of the Szegő polynomials are strictly inside the unit circle and all recursion coefficients γ_j are of magnitude smaller than one; see, e.g., [1,16]. The leading coefficient of ϕ_j is $1/\delta_j$.

The first step in the new zerofinders of this paper is to determine recursion coefficients $\{\gamma_j\}_{j=1}^n$, such that the Szegő polynomial ϕ_n satisfies

$$\delta_n \phi_n(\zeta) = \eta_1^n \psi_n(z), \tag{7}$$

where

$$\zeta = \eta_1 z + \eta_2, \tag{8}$$

and the constants η_1 and η_2 are chosen so that the zeros z_j of ψ_n are mapped to zeros ζ_j of ϕ_n inside the unit circle. We refer to this change of variable as a *rescaling* of the monic polynomial $\psi_n(z)$. Its construction is discussed in Section 2. Thus, the problem of determining the zeros of ψ_n is reduced to the problem of computing the zeros of a Szegő polynomial of degree n. Section 3 considers two methods for this purpose, based on a matrix formulation of the recursion relation (4). This gives an $n \times n$ upper Hessenberg matrix whose eigenvalues are the zeros of ϕ_n. We refer to this matrix, which is described in [11], as the *Szegő–Hessenberg matrix* associated with ϕ_n. Having computed the eigenvalues ζ_j of this matrix, we use the relation (8) to compute the zeros z_j of ψ_n.

A third method for computing the zeros of $\psi_n(z)$ is to use the power-basis coefficients of the monic Szegő polynomial $\Phi_n(\zeta) := \delta_n \phi_n(\zeta)$ of (7) to form the companion matrix associated with Φ_n, compute its eigenvalues, and transform these back to the z-variable using (8). In other words, to use the companion matrix of the rescaled monic polynomial Φ_n instead of that of ψ_n. This method is included in the numerical results we report in Section 4.

Section 4 compares the use of the QR algorithm with balancing for computing the eigenvalues of the Szegő–Hessenberg, the companion matrix (2) of ψ_n, and the companion matrix of the rescaled polynomial Φ_n. We note in passing that these are all upper Hessenberg matrices. Balancing is commonly used for improving the accuracy of the computed eigenvalues; see [7] for a discussion on balancing of the companion matrix. In our experiments we found that when the parameters η_1 and η_2 for the rescaling are chosen so that all zeros of ϕ_n are inside the unit circle and one zero is close to the unit circle, the computed eigenvalues of the Szegő–Hessenberg matrix and of the companion matrix of the rescaled polynomial (7) generally provide more accurate zeros of ψ_n than those of the companion matrix of ψ_n. This rescaling is achieved by application of the Schur–Cohn test as described in Section 3. Numerous computed examples, some of which are reported in Section 4, indicate that computing eigenvalues of the Szegő–Hessenberg matrix after balancing often gives the zeros of ψ_n with higher accuracy than computing eigenvalues of the companion matrix of the scaled polynomial (7) after balancing. Both methods, in general, give higher accuracy in the computed zeros than computing the zeros of ψ_n as eigenvalues of the balanced companion matrix.

The other zerofinder for Szegő polynomials discussed in Section 3 is the continuation method previously introduced in [2]. For many polynomials ψ_n, this method yields higher accuracy than the computation of the eigenvalues of the associated companion or Szegő–Hessenberg matrices. Section 4 presents numerical examples and Section 5 contains concluding remarks.

2. Computation of Szegő polynomials

Given a polynomial $\psi_n(z)$ in power-basis form (1), we compute the recursion coefficients $\{\gamma_j\}_{j=1}^n$ of the family of Szegő polynomials $\{\phi_j\}_{j=0}^n$, chosen so that ϕ_n satisfies (7), by first transforming the polynomial ψ_n so that the average of its zeros vanishes. Then we determine a disk centered at the origin that contains all zeros of the transformed polynomial. The complex plane is then scaled so that this disk becomes the unit disk. In this fashion, the problem of determining the zeros of the polynomial ψ_n has been transformed into an equivalent problem of determining the zeros of a polynomial with all zeros in the unit disk. We may assume that the latter polynomial has leading coefficient one, and identify it with the monic Szegő polynomial $\Phi_n = \delta_n\phi_n$. Given the power-basis coefficients of Φ_n, the recursion coefficients of the family of Szegő polynomials $\{\phi_j\}_{j=0}^n$ can be computed by the Schur–Cohn algorithm. The remainder of this section describes details of the computations outlined.

Let $\{z_j\}_{j=1}^n$ denote the zeros of ψ_n and introduce the average of the zeros

$$\rho := \frac{1}{n}\sum_{j=1}^{n} z_j. \tag{9}$$

We evaluate this quantity as $\rho = -\alpha_{n-1}/n$, and define the new variable $\hat{z} = z - \rho$. The polynomial $\hat{\psi}_n(\hat{z}) := \psi_n(z)$ can be written as

$$\hat{\psi}_n(\hat{z}) = \hat{z}^n + \hat{\alpha}_{n-2}\hat{z}^{n-2} + \cdots + \hat{\alpha}_1\hat{z} + \hat{\alpha}_0. \tag{10}$$

The coefficients $\{\hat{\alpha}_j\}_{j=0}^{n-2}$ can be computed from the coefficients $\{\alpha_j\}_{j=0}^{n-1}$ in $\mathcal{O}(n^2)$ arithmetic operations.

We now scale the $\hat{z}$-plane in two steps in order to move the zeros of $\hat{\psi}_n$ inside the unit circle. Our choice of scaling is motivated by the following result mentioned by Ostrowski [23].

Proposition 2.1. *Let χ_n be a polynomial of degree n of the form*

$$\chi_n(z) = z^n + \beta_{n-2}z^{n-2} + \cdots + \beta_1 z + \beta_0, \tag{11}$$

and assume that

$$\max_{0\leqslant j\leqslant n-2} |\beta_j| = 1.$$

Then all zeros of χ_n are contained in the open disk $\{z: |z| < \frac{1}{2}(1+\sqrt{5})\}$ in the complex plane.

Proof. Let z be a zero of χ_n and assume that $|z| > 1$. Then

$$z^n = -\beta_{n-2}z^{n-2} - \cdots - \beta_1 z - \beta_0,$$

and it follows that

$$|z|^n \leqslant \sum_{j=0}^{n-2} |z|^j = \frac{|z|^{n-1} - 1}{|z| - 1}.$$

This inequality can be written as

$$|z|^{n-1}(|z|^2 - |z| - 1) \leqslant -1. \tag{12}$$

Since $|z|^2-|z|-1=(|z|-\frac{1}{2}(1-\sqrt{5}))(|z|-\frac{1}{2}(1+\sqrt{5}))$, inequality (12) can only hold for $|z| < \frac{1}{2}(1+\sqrt{5})$. □

After the change of variable $\tilde{z}:=\sigma\hat{z}$, where $\sigma > 0$ is chosen so that

$$\max_{2\leqslant j\leqslant n} \sigma^j|\hat{\alpha}_{n-j}| = 1,$$

the polynomial $\tilde{\psi}_n(\tilde{z}):=\sigma^n\hat{\psi}_n(\hat{z})$ satisfies the conditions of the proposition.

Define the scaling factor

$$\tau:=\frac{2}{1+\sqrt{5}}. \tag{13}$$

By Proposition 2.1 the change of variables

$$\zeta:=\tau\tilde{z} \tag{14}$$

yields a monic polynomial

$$\Phi_n^{(\tau)}(\zeta):=\tau^n\tilde{\psi}_n(\tilde{z}) \tag{15}$$

with all zeros inside the unit circle.

We identify $\Phi_n^{(\tau)}$ with the monic Szegő polynomial $\delta_n\phi_n$, and wish to compute the recursion coefficients $\{\gamma_j\}_{j=1}^n$ that determine polynomials of lower degree $\{\phi_j\}_{j=0}^{n-1}$ in the same family of Szegő polynomials; see (4). This can be done by using the relationship between the coefficients of ϕ_j in power form and the coefficients of the associated auxiliary polynomial. Specifically, it follows from (6) that if

$$\phi_j(z) = \sum_{k=0}^{j} \beta_{j,k}z^k, \tag{16}$$

then

$$\phi_j^*(z) = \sum_{k=0}^{j} \bar{\beta}_{j,k-j}z^k.$$

Thus, given the Szegő polynomial ϕ_n in power form, we can determine the coefficients of the associated auxiliary polynomial ϕ_n^* in power form and apply the recursion formula (4) "backwards" in order to determine the recursion coefficient γ_n and the coefficients of the polynomials ϕ_{n-1} and ϕ_{n-1}^* in power form. In this manner we can determine the recursion coefficients γ_j for decreasing values of the index j.

The Schur–Cohn algorithm, see, e.g., Henrici [17, Chapter 6], is an implementation of these computations. The algorithm requires $\mathcal{O}(n^2)$ arithmetic operations to determine the recursion coefficients $\{\gamma_j\}_{j=1}^n$ from the representation of ϕ_n in power form (16).

We remark that the Schur–Cohn algorithm is known for its use in determining whether a given polynomial, in power form, has all zeros inside the unit circle. In this context it is known as the Schur–Cohn test; see [17, Chapter 6]. All zeros being strictly inside the unit circle is equivalent with all recursion coefficients $\{\gamma_j\}_{j=1}^n$ being of magnitude strictly smaller than one. We will return to this property of the recursion coefficients in Section 3.

Perhaps the first application of the Schur–Cohn algorithm to the computation of zeros of polynomials was described by Lehmer [19], who covered the complex plane by disks and used the Schur–Cohn test to determine which disks contain zeros of the polynomial. Lehmer's method can be viewed as a generalization of the bisection method to the complex plane. It is discussed in [17, Chapter 6].

3. The zerofinders

We present two zerofinders for ϕ_n and assume that the recursion coefficients $\{\gamma_j\}_{j=1}^n$ as well as the auxiliary coefficients $\{\sigma_j\}_{j=1}^n$ are available.

3.1. An eigenvalue method

Eliminating the auxiliary polynomials ϕ_j^* in the recursion formula (4) yields an expression for ϕ_{j+1} in terms of Szegő polynomials of lower degree. Writing the expressions for the first $n+1$ Szegő polynomials in matrix form yields

$$[\phi_0(z), \phi_1(z), \dots, \phi_{n-1}(z)]H_n = z[\phi_0(z), \phi_1(z), \dots, \phi_{n-1}(z)] - [0, \dots, 0, \phi_n(z)], \tag{17}$$

where

$$H_n = \begin{bmatrix} -\gamma_1 & -\sigma_1\gamma_2 & -\sigma_1\sigma_2\gamma_3 & \cdots & & -\sigma_1\cdots\sigma_{n-1}\gamma_n \\ \sigma_1 & -\bar{\gamma}_1\gamma_2 & -\bar{\gamma}_1\sigma_2\gamma_3 & \cdots & & -\bar{\gamma}_1\sigma_2\cdots\sigma_{n-1}\gamma_n \\ & \sigma_2 & -\bar{\gamma}_2\gamma_3 & \cdots & & -\bar{\gamma}_2\sigma_3\cdots\sigma_{n-1}\gamma_n \\ & & \ddots & & & \vdots \\ & & \sigma_{n-2} & & -\bar{\gamma}_{n-2}\gamma_{n-1} & -\bar{\gamma}_{n-2}\sigma_{n-1}\gamma_n \\ 0 & & & & \sigma_{n-1} & -\bar{\gamma}_{n-1}\gamma_n \end{bmatrix} \in \mathbb{C}^{n\times n} \tag{18}$$

is the Szegő–Hessenberg matrix associated with the Szegő polynomials $\{\phi_j\}_{j=0}^n$; see [11]. Eq. (17) shows that the eigenvalues of the upper Hessenberg matrix H_n are the zeros of ϕ_n. Thus, we can compute the zeros of ϕ_n by determining the eigenvalues of H_n.

Let ζ_j, $1 \leqslant j \leqslant n$, denote the zeros of ϕ_n. The scaling parameters η_1 and η_2 in (8) are chosen so that all zeros of ϕ_n are inside the unit circle. However, for some polynomials ψ_n, the scaling may be such that

$$\kappa_n := \max_{1\leqslant j\leqslant n} |\zeta_j| \ll 1.$$

We have noticed that we can determine the zeros of ψ_n with higher accuracy when the disk is rescaled to make κ_n close to one. Such a rescaling is easy to achieve by repeated application of the Schur–Cohn test as follows. Instead of scaling $\tilde{z}$ by the factor (13) in (14), we scale $\tilde{z}$ by $\tau := \sqrt{2}/(1+\sqrt{5})$ and apply the Schur–Cohn test to determine whether all zeros of the scaled polynomial (15) so obtained are inside the unit circle. If they are not, then we increase the scaling factor τ in (14) by a factor $\Delta\tau := (2/(1+\sqrt{5}))^{1/10}$ and check whether the (re)scaled polynomial (15) obtained has all zeros inside the unit circle. The scaling factor τ is increased repeatedly by the factor $\Delta\tau$ until the polynomial (15) has all its zeros inside the unit circle. On the other hand, if the polynomial (15) associated with the scaling factor $\tau = \sqrt{2}/(1+\sqrt{5})$ has all zeros inside the unit circle, we repeatedly decrease τ by a factor $(\Delta\tau)^{-1}$ until a scaling factor τ has been determined, such that all zeros of the polynomial $\Phi_n^{(\tau)}$ are inside the unit disk, but at least one zero of $\Phi_n^{(\tau/\Delta\tau)}$ is not. Our choice of scaling factor τ in (14) assures that the monic polynomial (15) has all its zeros inside the unit circle and (at least) one zero close to the unit circle.

The scaling factors τ in (14) for the computed examples reported in Section 4 have been determined as described above. In our experience, the time spent rescaling the disk is negligible compared to the time required to compute the eigenvalues of H_n, because each rescaling only requires $\mathcal{O}(n^2)$ arithmetic operations.

After determining the scaling factor τ as described above and computing the recursion coefficients $\{\gamma_j\}_{j=1}^n$ via the Schur–Cohn test, we form the Szegő–Hessenberg matrix (18), balance it, and compute its eigenvalues using the QR algorithm.

3.2. A continuation method

Similarly as in the method described in Section 3.1, we first determine the recursion coefficients of the Szegő polynomials $\{\phi_j\}_{j=0}^n$ such that Eq. (7) holds, as described above. We then apply the continuation method for computing zeros of Szegő polynomials developed in [2]. In this method the Szegő–Hessenberg matrix (18) is considered a function of the last recursion parameter γ_n. Denote this parameter by $t \in \mathbb{C}$ and the associated Szegő–Hessenberg matrix by $H_n(t)$. Thus, we write the matrix (18) as $H_n(\gamma_n)$. When $|t| = 1$, the Szegő–Hessenberg matrix $H_n(t)$ is unitary. Assume that $\gamma_n \neq 0$. Then $H_n(\gamma_n/|\gamma_n|)$ is the closest unitary matrix to $H_n(\gamma_n)$; see [2] for details. The continuation method for computing zeros of Szegő polynomials consists of the following steps:

(i) Compute the eigenvalues of the unitary upper Hessenberg matrix $H_n(\gamma_n/|\gamma_n|)$.

(ii) Apply a continuation method for tracking the path of each eigenvalue of the matrix $H_n(t)$ as t is moved from $\gamma_n/|\gamma_n|$ to γ_n.

Several algorithms that require only $\mathcal{O}(n^2)$ arithmetic operations for the computations of Step (i) are available; see, e.g. [4–6,12–15]. If the coefficients α_j in (1) are real, then the method discussed in [3] can also be applied. These methods compute the eigenvalues of $H_n(\gamma_n/|\gamma_n|)$ without explicitly forming the matrix elements. In the numerical experiments reported in Section 4, we used the implementation [4,5] of the divide-and-conquer method described in [14,15]. The computations required for this method can readily be implemented on a parallel computer. This may be of importance in the application of the zerofinder in real-time filter design; see, e.g., Parks and Burrus [24] and references therein for more on this application of polynomial zerofinders.

We have found that for many polynomials ψ_n, the continuation method determines the zeros with higher accuracy than the method discussed in Section 3.1. The continuation method determines the zeros of the Szegő polynomial ϕ_n close to the unit circle particularly rapidly. However, our present implementation of the continuation method may fail to determine all zeros for some polynomials ψ_n when the pathfollowing is complicated by (numerous) bifurcation points. These cases are easy to identify; see [2] for a discussion and remedies.

Table 1
Ten polynomials of degree $n = 15$ with zeros in D_1

Differences:	CB	SHB	CM	CBS
	6.67E−05	4.89E−06	4.57E−06	6.82E−06
	1.66E−03	7.57E−05	5.49E−05	2.11E−04
	1.20E−01	3.06E−03	—	1.83E−02
	8.41E−04	2.45E−05	3.91E−05	6.22E−04
	9.66E−04	5.88E−05	5.82E−05	1.51E−04
	2.75E−05	5.20E−07	1.79E−07	2.40E−06
	3.34E−05	5.75E−06	2.71E−07	2.05E−05
	1.67E−05	2.85E−06	2.25E−06	5.52E−05
	2.72E−04	6.60E−06	7.48E−07	3.77E−05
	7.60E−05	1.16E−06	7.40E−07	3.30E−06
Averages:	1.24E−02	3.24E−04	1.79E−05	1.94E−03

Residuals:	CB	SHB	CM	CBS	ψ_n
	3.85E−06	9.06E−07	4.89E−07	1.10E−06	6.94E−07
	3.31E−07	9.68E−08	2.05E−08	1.15E−07	1.47E−08
	3.16E−05	1.30E−05	—	2.41E−05	5.80E−07
	2.48E−06	9.15E−07	3.16E−07	1.47E−06	6.62E−08
	5.24E−06	6.74E−07	1.18E−06	1.50E−06	3.58E−07
	8.64E−08	2.13E−08	1.47E−08	4.12E−08	2.18E−09
	1.87E−06	6.88E−07	5.66E−07	8.80E−07	2.92E−08
	2.93E−06	2.48E−06	2.76E−07	2.71E−06	4.34E−08
	2.14E−07	7.87E−08	6.35E−08	3.23E−08	6.32E−09
	1.07E−06	4.44E−07	9.72E−08	9.11E−07	2.11E−08
Averages:	4.97E−06	1.93E−06	3.36E−07	3.28E−06	1.82E−07

	Differences				Residuals			
CB	0				0			
SHB	10	2			10	2		
CM	9	8	8		9	8	7	
CBS	9	0	1	0	10	1	2	1

We remark that other continuation methods also are available, such as the method proposed by Li and Zeng [20] for computing the eigenvalues of a general Hessenberg matrix. This method does not use the structure of the Hessenberg matrices (18), i.e., the fact that the last recursion coefficient γ_n is a natural continuation parameter. However, it may be possible to apply some techniques developed in [20] to improve the performance of the continuation method of this paper; see [2] for a discussion and references to other continuation methods.