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Differential Equations & Linear Algebra

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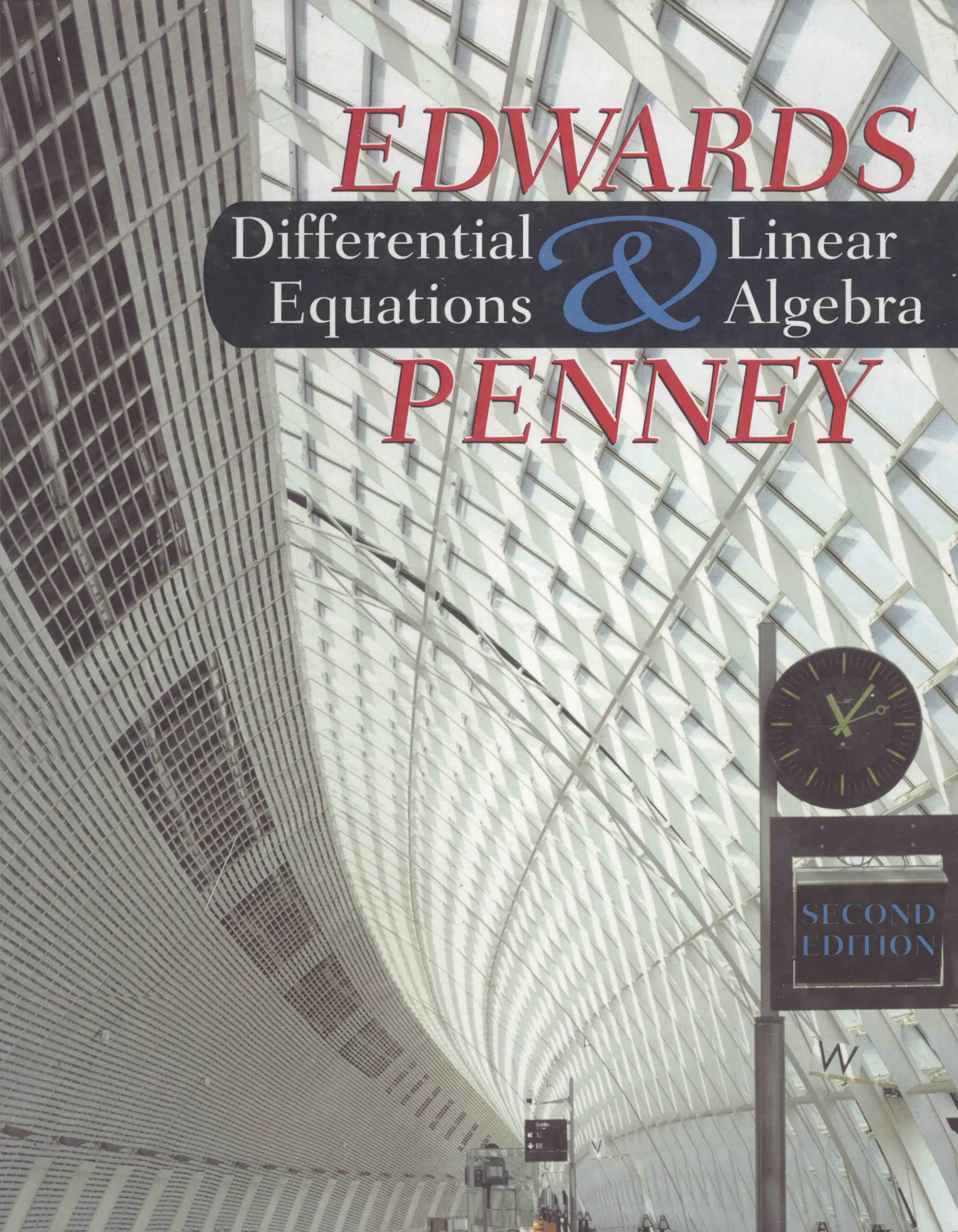
SECOND EDITION



DIFFERENTIAL EQUATIONS & LINEAR ALGEBRA

Second Edition

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DIFFERENTIAL EQUATIONS & LINEAR ALGEBRA

APPLICATION MODULES

The modules listed here follow the indicated sections in the text. Most provide computing projects that illustrate the content of the corresponding text sections. *Maple*, *Mathematica*, and MATLAB versions of these investigations are included in the Applications Manual that accompanies this text.

PREFACE

Many introductory differential equations courses in the recent past have emphasized the formal solution of standard types of differential equations using a (seeming) grab-bag of systematic solution techniques. Many students have concentrated on learning to match memorized methods with memorized equations. The evolution of the present text is based on experience teaching a course with a greater emphasis on conceptual ideas and the use of applications and computing projects to involve students in more intense and sustained problem-solving experiences.

Both the conceptual and the computational aspects of such a course depend heavily on the perspective and techniques of linear algebra. Consequently, the study of differential equations and linear algebra in tandem reinforces the learning of both subjects. In this book we therefore have combined core topics in elementary differential equations with those concepts and methods of elementary linear algebra that are needed for a contemporary introduction to differential equations.

The availability of technical computing environments like *Maple*, *Mathematica*, and MATLAB is reshaping the role and applications of differential equations in science and engineering, and has shaped our approach in this text. New technology motivates a shift in emphasis from traditional manual methods to both qualitative and computer-based methods that

- render accessible a wider range of more realistic applications;
- permit the use of both numerical computation and graphical visualization to develop greater conceptual understanding; and
- encourage empirical investigations that involve deeper thought and analysis than standard textbook problems.

Major Features

The following features of this text are intended to support a contemporary differential equations course with linear algebra that augments traditional core skills with conceptual perspectives that students will need for the effective use of differential equations in their subsequent work and study:

- The organization of the book emphasizes linear systems of algebraic and differential equations. Chapter 3 introduces matrices and determinants as needed for concrete computational purposes. Chapter 4 introduces vector spaces in preparation for understanding (in Chapter 5) the solution set of an nth order homogeneous linear differential equation as an n-dimensional vector space of functions, and for realizing that finding a general solution of the equation amounts to finding a basis for its solution space. (Students who proceed to a subsequent course in abstract linear algebra may benefit especially from this concrete experience with vector spaces.) Chapter 6 introduces eigenvalues and eigenvectors in preparation for solving linear systems of differential equations in Chapters 7 and 8. In Chapter 8 we may go a bit further than usual with the computation of matrix exponentials. These linear tools are applied to the analysis of nonlinear systems and phenomena in Chapter 9.

- Coverage of seldom-used topics has been trimmed and new topics added to place a greater emphasis on core techniques as well as qualitative aspects of the subject associated with direction fields, solution curves, phase plane portraits, and dynamical systems. We combine symbolic, graphic, and numeric solution methods wherever it seems advantageous. A fresh computational flavor should be evident in figures, examples, problems, and applications throughout the text. Over 10% of the examples in the text are new or newly revised for this edition.
- The first course in differential equations should also be a window on the world of mathematics. While it is neither feasible nor desirable to include proofs of the fundamental existence and uniqueness theorems along the way in an elementary course, students need to see precise and clear-cut statements of these theorems, and to understand their role in the subject. We include appropriate existence and uniqueness proofs in Appendix A, and occasionally refer to them in the main body of the text.
- While our approach reflects the widespread use of new computer methods for the solution of differential equations and linear systems, certain elementary analytical methods of solution (as in Chapters 1 and 5) are important for students to learn. Effective and reliable use of numerical methods often requires preliminary analysis using standard elementary techniques; the construction of a realistic numerical model often is based on the study of a simpler analytical model. We therefore continue to stress the mastery of traditional solution techniques (especially through the inclusion of extensive problem sets).

Computing Features

The following features highlight the flavor of computing technology that distinguishes much of our exposition.

- Almost 600 *computer-generated figures*—over half of them new for this edition and most constructed using *Mathematica* or MATLAB—show students vivid pictures of direction fields, solution curves, and phase plane portraits that bring symbolic solutions of differential equations to life.
- About 3 dozen *application modules* follow key sections throughout the text. Most of these applications outline "technology neutral" investigations illustrating the use of technical computing systems and seek to actively engage students in the application of new technology.
- A fresh *numerical emphasis* that is afforded by the early introduction of numerical solution techniques in Chapter 2 (on mathematical models and numerical methods). Here and in Section 7.6, where numerical techniques for systems are treated, a concrete and tangible flavor is achieved by the inclusion of numerical algorithms presented in parallel fashion for systems ranging from graphing calculators to MATLAB.
- A *conceptual perspective* shaped by the availability of computational aids, which permits a leaner and more streamlined coverage of certain traditional manual topics (like exact equations and variation of parameters) in Chapters 1 and 5.

Applications

Mathematical modeling is a goal and constant motivation for the study of differential equations. To sample the range of applications in this text, take a look at the following questions:

- What explains the commonly observed time lag between indoor and outdoor daily temperature oscillations? (Section 1.5)
- What makes the difference between doomsday and extinction in alligator populations? (Section 2.1)
- How do a unicycle and a two-axle car react differently to road bumps? (Sections 5.6 and 7.4)
- Why might an earthquake demolish one building and leave standing the one next door? (Section 7.4)
- How can you predict the time of next perihelion passage of a newly observed comet? (Section 7.6)
- What determines whether two species will live harmoniously together, or whether competition will result in the extinction of one of them and the survival of the other? (Section 9.3)

Organization and Content

We have reshaped the usual approach and sequence of topics to accommodate new technology and new perspectives. For instance:

- After a precis of first-order equations in Chapter 1 (though with the coverage of certain traditional symbolic methods streamlined a bit), Chapter 2 offers an early introduction to mathematical modeling, stability and qualitative properties of differential equations, and numerical methods—a combination of topics that frequently are dispersed later in an introductory course.
- Chapters 3 (Linear Systems and Matrices), 4 (Vector Spaces), and 6 (Eigenvalues and Eigenvectors) provide concrete and self-contained coverage of the elementary linear algebra concepts and techniques that are needed for the solution of linear differential equations and systems. Chapter 4 now includes the new sections 4.5 (row and column spaces) and 4.6 (orthogonal vectors in $\mathbf{R}^n$) that have been added for this edition. Chapter 6 concludes with applications of diagonalizable matrices and a proof of the Cayley-Hamilton theorem for such matrices.
- Chapter 5 exploits the linear algebra of Chapters 3 and 4 to present efficiently the theory and solution of single linear differential equations. Chapter 7 is based on the eigenvalue approach to linear systems, and includes (in Section 7.5) the Jordan normal form for matrices and its application to the general Cayley-Hamilton theorem. This chapter includes an unusual number of applications (ranging from railway cars to earthquakes) of the various cases of the eigenvalue method, and concludes in Section 7.6 with numerical methods for systems.
- Chapter 8 is devoted to matrix exponentials with applications to linear systems of differential equations. The spectral decomposition method of Section 8.3 offers students an especially concrete approach to the computation of matrix exponentials. Our treatment of this material owes much to advice and

course notes provided by Professor Dar-Veig Ho of the Georgia Institute of Technology.

- Chapter 9 exploits linear methods for the investigation of nonlinear systems and phenomena, and ranges from phase plane analysis to applications involving ecological and mechanical systems.
- Chapters 10 treats Laplace transform methods for the solution of constant-coefficient linear differential equations with a goal of handling the piecewise continuous and periodic forcing functions that are common in physical applications. Chapter 11 treats power series methods with a goal of discussing Bessel's equation with sufficient detail for the most common elementary applications.

Problems, Applications, and Solutions Manuals

Over 15% of the text's almost 2300 problems are new for this edition or are newly revised to include graphic or qualitative content. Accordingly, the answer section now includes almost 200 new computer-generated figures illustrating those which students are expected to construct.

The answer section for this revision has been expanded considerably to increase its value as a learning aid. It now includes the answers to most odd-numbered problems plus a good many even-numbered ones. The 590-page **Instructor's Solutions Manual** (0-13-148148-7) accompanying this book provides worked-out solutions for most of the problems in the book, and the 330-page **Student Solutions Manual** (0-13-148251-3) contains solutions for most of the odd-numbered problems.

The approximately 3 dozen application modules in the text contain additional problem and project material designed largely to engage students in the exploration and application of computational technology. These investigations are expanded considerably in the 250-page **Applications Manual** (0-13148148-7) that accompanies the text and supplements it with additional and sometimes more challenging investigations. Each section in this manual has parallel subsections **Using *Maple***, **Using *Mathematica***, and **Using MATLAB** that detail the applicable methods and techniques of each system, and will afford student users an opportunity to compare the merits and styles of different computational systems.

Technology Manuals and Website

The author-written solutions and applications manuals described above, as well as the additional technology manuals listed below, are available shrink-wrapped free with the textbook upon order using the indicated ISBN numbers:

- Text with *Student Solutions Manual* (0-13-152360-0)
- Text with *Applications Manual* (0-13-151944-1)
- Text with David Calvis, *Mathematica for Differential Equations: Projects, Insights, Syntax, and Animations* (0-13-161732-X)
- Text with Selwyn Hollis, *A Mathematica Companion for Differential Equations* (0-13-151903-4)
- Text with Robert Gilbert & George Hsiao, *Maple Projects for Differential Equations* (0-13-161734-6)
- Text with John Polking & David Arnold, *Ordinary Differential Equations Using* MATLAB, 3rd edition (0-13-151905-0)

Notebooks and worksheets supporting these manuals—plus additional software including a package of *Maple* worksheets keyed to this text by John Maloney—are available for downloading at the website **www.prenhall.com/edwards**. Many of the figures in this text were computer generated using Polking's MATLAB programs **dfield** and **pplane** that are linked at the site. Another MATLAB-based ODE package that has impressive graphical capabilities and is referenced in the text is **Iode** (see **www.math.uiuc.edu/iode**).

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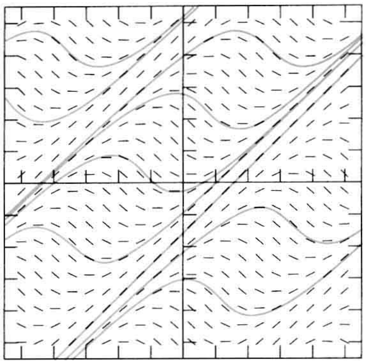

CHAPTER 1

First-Order Differential Equations

1.1 Differential Equations and Mathematical Models

The laws of the universe are written in the language of mathematics. Algebra is sufficient to solve many static problems, but the most interesting natural phenomena involve change and are described by equations that relate changing quantities.

Because the derivative $dx/dt = f'(t)$ of the function f is the rate at which the quantity $x = f(t)$ is changing with respect to the independent variable t, it is natural that equations involving derivatives are frequently used to describe the changing universe. An equation relating an unknown function and one or more of its derivatives is called a **differential equation**.

Example 1 The differential equation

$$\frac{dx}{dt} = x^2 + t^2$$

involves both the unknown function $x(t)$ and its first derivative $x'(t) = dx/dt$. The differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 7y = 0$$

involves the unknown function y of the independent variable x and the first two derivatives y' and y'' of y. ■

The study of differential equations has three principal goals:

1. To discover the differential equation that describes a specified physical situation.
2. To find—either exactly or approximately—the appropriate solution of that equation.
3. To interpret the solution that is found.

In algebra, we typically seek the unknown *numbers* that satisfy an equation such as $x^3+7x^2-11x+41=0$. By contrast, in solving a differential equation, we are challenged to find the unknown *functions* $y=y(x)$ for which an identity such as $y'(x)=2xy(x)$—that is, the differential equation

$$\frac{dy}{dx}=2xy$$

—holds on some interval of real numbers. Ordinarily, we will want to find *all* solutions of the differential equation, if possible.

Example 2 If C is a constant and

$$y(x)=Ce^{x^2}, \tag{1}$$

then

$$\frac{dy}{dx}=C\left(2xe^{x^2}\right)=(2x)\left(Ce^{x^2}\right)=2xy.$$

Thus every function $y(x)$ of the form in Eq. (1) *satisfies*—and thus is a solution of—the differential equation

$$\frac{dy}{dx}=2xy \tag{2}$$

for all x. In particular, Eq. (1) defines an *infinite* family of different solutions of this differential equation, one for each choice of the arbitrary constant C. By the method of separation of variables (Section 1.4) it can be shown that every solution of the differential equation in (2) is of the form in Eq. (1). ■

Differential Equations and Mathematical Models

The following three examples illustrate the process of translating scientific laws and principles into differential equations. In each of these examples the independent variable is time t, but we will see numerous examples in which some quantity other than time is the independent variable.

Example 3 Newton's law of cooling may be stated in this way: The *time rate of change* (the rate of change with respect to time t) of the temperature $T(t)$ of a body is proportional to the difference between T and the temperature A of the surrounding medium (Fig. 1.1.1). That is,

$$\frac{dT}{dt}=-k(T-A), \tag{3}$$

where k is a positive constant. Observe that if $T>A$, then $dT/dt<0$, so the temperature is a decreasing function of t and the body is cooling. But if $T<A$, then $dT/dt>0$, so that T is increasing.

Thus the physical law is translated into a differential equation. If we are given the values of k and A, we should be able to find an explicit formula for $T(t)$, and then—with the aid of this formula—we can predict the future temperature of the body. ■

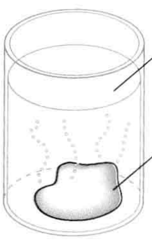

FIGURE 1.1.1. Newton's law of cooling, Eq. (3), describes the cooling of a hot rock in water.

Example 4 Torricelli's law implies that the *time rate of change* of the volume V of water in a draining tank (Fig. 1.1.2) is proportional to the square root of the depth y of water in the tank:

$$\frac{dV}{dt}=-k\sqrt{y}, \tag{4}$$

where k is a constant. If the tank is a cylinder with vertical sides and cross-sectional area A, then $V = Ay$, so $dV/dt = A \cdot (dy/dt)$. In this case Eq. (4) takes the form

$$\frac{dy}{dt} = -h\sqrt{y}, \qquad (5)$$

where $h = k/A$ is a constant. ■

Example 5

The *time rate of change* of a population $P(t)$ with constant birth and death rates is, in many simple cases, proportional to the size of the population. That is,

$$\frac{dP}{dt} = kP, \qquad (6)$$

where k is the constant of proportionality. ■

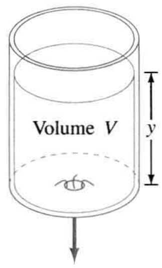

FIGURE 1.1.2. Torricelli's law of draining, Eq. (4), describes the draining of a water tank.

Let us discuss Example 5 further. Note first that each function of the form

$$P(t) = Ce^{kt} \qquad (7)$$

is a solution of the differential equation

$$\frac{dP}{dt} = kP$$

in (6). We verify this assertion as follows:

$$P'(t) = Cke^{kt} = k\left(Ce^{kt}\right) = kP(t)$$

for all real numbers t. Because substitution of each function of the form given in (7) into Eq. (6) produces an identity, all such functions are solutions of Eq. (6).

Thus, even if the value of the constant k is known, the differential equation $dP/dt = kP$ has *infinitely many* different solutions of the form $P(t) = Ce^{kt}$, one for each choice of the "arbitrary" constant C. This is typical of differential equations. It is also fortunate, because it may allow us to use additional information to select from among all these solutions a particular one that fits the situation under study.

Example 6

Suppose that $P(t) = Ce^{kt}$ is the population of a colony of bacteria at time t, that the population at time $t = 0$ (hours, h) was 1000, and that the population doubled after 1 h. This additional information about $P(t)$ yields the following equations:

$$1000 = P(0) = Ce^0 = C,$$
$$2000 = P(1) = Ce^k.$$

It follows that $C = 1000$ and that $e^k = 2$, so $k = \ln 2 \approx 0.693147$. With this value of k the differential equation in (6) is

$$\frac{dP}{dt} = (\ln 2)P \approx (0.693147)P.$$

Substitution of $k = \ln 2$ and $C = 1000$ in Eq. (7) yields the particular solution

$$P(t) = 1000e^{(\ln 2)t} = 1000(e^{\ln 2})^t = 1000 \cdot 2^t \qquad (\text{because } e^{\ln 2} = 2)$$

that satisfies the given conditions. We can use this particular solution to predict future populations of the bacteria colony. For instance, the predicted number of bacteria in the population after one and a half hours (when $t = 1.5$) is

$$P(1.5) = 1000 \cdot 2^{3/2} \approx 2828.$$

■