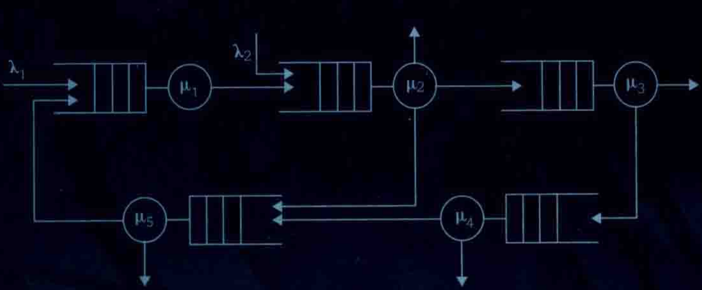
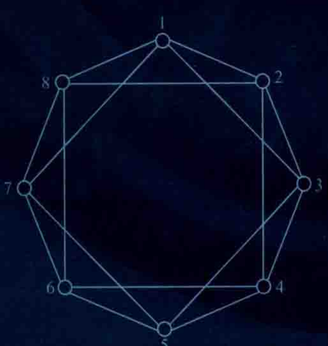
FUNDAMENTALS OF STOCHASTIC NETWORKS

Oliver C. Ibe





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PREFACE

This book brings into one volume two network models that can be broadly classified as *queueing network models* and *graphical network models*. Queueing networks are systems where customers move among service stations where they receive service. Usually, the service times and the order in which customers visit the service stations are random. The order in which service is received at the service stations is governed by a probabilistic routing schedule. Queueing networks are popularly used in traffic modeling in computer and telecommunications networks, transportation systems, and manufacturing networks. Graphical models are systems that use graphs to model different types of problems. They include Bayesian networks, which are also called *directed graphical models*, Boolean networks, and random networks. Graphical models are used in statistics, data mining, and social networks.

The need for a book of this nature arises from the fact that we live in an era of interdisciplinary studies and research activities when both networks are becoming important in areas that they were not originally used. Thus, any person involved in such interdisciplinary studies or research activities needs to have a good understanding of both types of networks. This book is intended to meet this need.

The book is organized into three parts. The first part, Chapters 1 and 2, deals with the basic concepts of probability (Chapter 1) and stochastic processes (Chapter 2). The second part, Chapters 3–6, deals with queueing systems. Specifically, Chapter 3 deals with basic queueing theory, particularly a class of queueing systems that we refer to as Markovian queueing systems. Chapter 4 deals with advanced queueing systems, particularly the non-Markovian queueing systems. Chapter 5 deals with queueing networks, and Chapter 6 deals with approximations of queueing networks. The third part, Chapters 7–10, deals

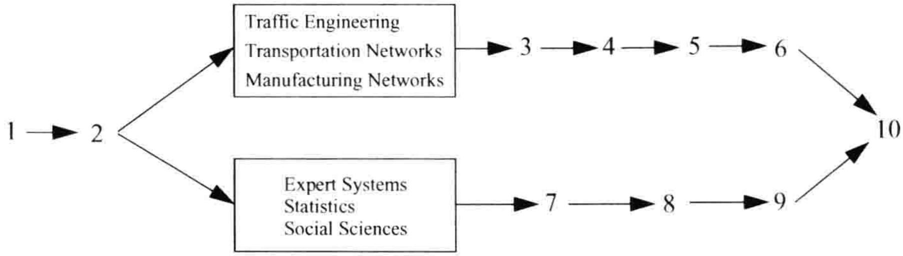

Figure 1 Precedence relations of chapters.

with graphical models. Chapter 7 deals with an introduction to graph theory, Chapter 8 deals with Bayesian networks, Chapter 9 deals with Boolean networks, and Chapter 10 deals with random networks.

The book is self-contained and is written with a view to circumventing the proof–theorem format that is traditionally used in stochastic systems modeling books. It is intended to be an introductory graduate text on stochastic networks and presents the basic results without much emphasis on proving theorems. Thus, it is designed for science and engineering applications. Students who have an interest in traffic engineering, transportation, and manufacturing networks will need to cover parts 1 and 2 as well as Chapter 10 in part 3, while students with an interest in expert systems, statistics, and social sciences will need to cover parts 1 and 3. The precedence relations among the chapters are shown in Figure 1.

ACKNOWLEDGMENTS

I would like to express my sincere gratitude to my wife, Christie, for bearing with my writing yet another book. She has been my greatest fan when it comes to my writing books. I would also like to acknowledge the encouraging words from our children Chidinma, Ogechi, Amanze, and Ugonna. I would like to express my sincere gratitude to my editor, Susanne Steitz-Filler, for her encouragement and for checking on me regularly to make sure that we met the deadlines. Finally, I would like to thank the anonymous reviewers for their useful comments and suggestions that helped to improve the quality of the book.

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Lowell, Massachusetts
January 2011

CONTENTS

1

BASIC CONCEPTS IN PROBABILITY

1.1 INTRODUCTION

The concepts of *experiments* and *events* are very important in the study of probability. In probability, an experiment is any process of trial and observation. An experiment whose outcome is uncertain before it is performed is called a *random* experiment. When we perform a random experiment, the collection of possible elementary outcomes is called the *sample space* of the experiment, which is usually denoted by Ω. We define these outcomes as elementary outcomes because exactly one of the outcomes occurs when the experiment is performed. The elementary outcomes of an experiment are called the *sample points* of the sample space and are denoted by w_i, $i = 1, 2, \ldots$ If there are n possible outcomes of an experiment, then the sample space is $\Omega = \{w_1, w_2, \ldots, w_n\}$. An *event* is the occurrence of either a prescribed outcome or any one of a number of possible outcomes of an experiment. Thus, an event is a subset of the sample space.

1.2 RANDOM VARIABLES

Consider a random experiment with sample space Ω. Let w be a sample point in Ω. We are interested in assigning a real number to each $w \in \Omega$. A random variable, $X(w)$, is a single-valued real function that assigns a real number,

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called the value of $X(w)$, to each sample point $w \in \Omega$. That is, it is a mapping of the sample space onto the real line.

Generally a random variable is represented by a single letter X instead of the function $X(w)$. Therefore, in the remainder of the book we use X to denote a random variable. The sample space Ω is called the *domain* of the random variable X. Also, the collection of all numbers that are values of X is called the *range* of the random variable X.

Let X be a random variable and x a fixed real value. Let the event A_x define the subset of Ω that consists of all real sample points to which the random variable X assigns the number x.

That is,

$$A_x = \{w | X(w) = x\} = [X = x].$$

Since A_x is an event, it will have a probability, which we define as follows:

$$p = P[A_x].$$

We can define other types of events in terms of a random variable. For fixed numbers x, a, and b, we can define the following:

$$[X \le x] = \{w | X(w) \le x\},$$
$$[X > x] = \{w | X(w) > x\},$$
$$[a < X < b] = \{w | a < X(w) < b\}.$$

These events have probabilities that are denoted by

- $P[X \le x]$ is the probability that X takes a value less than or equal to x.
- $P[X > x]$ is the probability that X takes a value greater than x; this is equal to $1 - P[X \le x]$.
- $P[a < X < b]$ is the probability that X takes a value that strictly lies between a and b.

1.2.1 Distribution Functions

Let X be a random variable and x be a number. As stated earlier, we can define the event $[X \le x] = \{x | X(w) \le x\}$. The distribution function (or the cumulative distribution function [CDF]) of X is defined by:

$$F_X(x) = P[X \le x] \quad -\infty < x < \infty.$$

That is, $F_X(x)$ denotes the probability that the random variable X takes on a value that is less than or equal to x. Some properties of $F_X(x)$ include:

1. $F_X(x)$ is a nondecreasing function, which means that if $x_1 < x_2$, then $F_X(x_1) \le F_X(x_2)$. Thus, $F_X(x)$ can increase or stay level, but it cannot go down.

2. $0 \le F_X(x) \le 1$
3. $F_X(\infty) = 1$
4. $F_X(-\infty) = 0$
5. $P[a < X \le b] = F_X(b) - F_X(a)$
6. $P[X > a] = 1 - P[X \le a] = 1 - F_X(a)$

1.2.2 Discrete Random Variables

A discrete random variable is a random variable that can take on at most a countable number of possible values. For a discrete random variable X, the *probability mass function* (PMF), $p_X(x)$, is defined as follows:

$$p_X(x) = P[X = x].$$

The PMF is nonzero for at most a countable or countably infinite number of values of x. In particular, if we assume that X can only assume one of the values $x_1, x_2, \ldots, x_n$, then:

$$\begin{aligned} p_X(x_i) &\ge 0 \quad i = 1, 2 \ldots, n, \\ p_X(x) &= 0 \quad \text{otherwise.} \end{aligned}$$

The CDF of X can be expressed in terms of $p_X(x)$ as follows:

$$F_X(x) = \sum_{k \le x} p_X(k).$$

The CDF of a discrete random variable is a step function. That is, if X takes on values $x_1, x_2, x_3, \ldots$, where $x_1 < x_2 < x_3 < \ldots$, then the value of $F_X(x)$ is constant in the interval between x_{i-1} and x_i and then takes a jump of size $p_X(x_i)$ at x_i, $i = 2, 3, \ldots$. Thus, in this case, $F_X(x)$ represents the sum of all the probability masses we have encountered as we move from $-\infty$ to x.

1.2.3 Continuous Random Variables

Discrete random variables have a set of possible values that are either finite or countably infinite. However, there exists another group of random variables that can assume an uncountable set of possible values. Such random variables are called continuous random variables. Thus, we define a random variable X to be a continuous random variable if there exists a nonnegative function $f_X(x)$, defined for all real $x \in (-\infty, \infty)$, having the property that for any set A of real numbers,

$$P[X \in A] = \int_A f_X(x)\,dx.$$

The function $f_X(x)$ is called the *probability density function* (PDF) of the random variable X and is defined by:

$$f_X(x) = \frac{dF_X(x)}{dx}.$$

The properties of $f_X(x)$ are as follows:

1. $f_X(x) \geq 0$
2. Since X must assume some value, $\int_{-\infty}^{\infty} f_X(x)dx = 1$
3. $P[a \leq X \leq b] = \int_a^b f_X(x)dx$, which means that $P[X = a] = \int_a^a f_X(x)dx = 0$. Thus, the probability that a continuous random variable will assume any fixed value is zero.
4. $P[X < a] = P[X \leq a] = F_X(a) = \int_{-\infty}^{a} f_X(x)dx$

1.2.4 Expectations

If X is a random variable, then the *expectation* (or *expected value* or *mean*) of X, denoted by E[X], is defined by:

$$E[X] = \begin{cases} \sum_i x_i p_X(x_i) & X \text{ discrete} \\ \int_{-\infty}^{\infty} x f_X(x)dx & X \text{ continuous} \end{cases}$$

Thus, the expected value of X is a weighted average of the possible values that X can take, where each value is weighted by the probability that X takes that value. The expected value of X is sometimes denoted by $\overline{X}$.

1.2.5 Moments of Random Variables and the Variance

The nth moment of the random variable X, denoted by $E[X^n] = \overline{X^n}$, is defined by:

$$E[X^n] = \overline{X^n} = \begin{cases} \sum_i x_i^n p_X(x_i) & X \text{ discrete} \\ \int_{-\infty}^{\infty} x^n f_X(x)dx & X \text{ continuous} \end{cases}$$

for $n = 1, 2, 3, \ldots$. The first moment, $E[X]$, is the expected value of X.

We can also define the *central moments* (or *moments about the mean*) of a random variable. These are the moments of the difference between a random variable and its expected value. The nth central moment is defined by

$$E\left[(X - \overline{X})^n\right] = \overline{(X - \overline{X})^n} = \begin{cases} \sum_i (x_i - \overline{X})^n p_X(x_i) & X \text{ discrete} \\ \int_{-\infty}^{\infty} (x - \overline{X})^n f_X(x)dx & X \text{ continuous} \end{cases}$$

The central moment for the case of $n = 2$ is very important and carries a special name, the *variance*, which is usually denoted by σ_X^2. Thus,

$$\sigma_X^2 = E\left[\left(X-\bar{X}\right)^2\right] = \overline{\left(X-\bar{X}\right)^2} = \begin{cases} \sum_i \left(x_i-\bar{X}\right)^2 p_X(x_i) & X \quad \text{discrete} \\ \int_{-\infty}^{\infty} \left(x-\bar{X}\right)^2 f_X(x)\,dx & X \quad \text{continuous} \end{cases}$$

1.3 TRANSFORM METHODS

Different types of transforms are used in science and engineering. In this book we consider two types of transforms: the z-transform of PMFs and the s-transform of PDFs of nonnegative random variables. These transforms are particularly used when random variables take only nonnegative values, which is usually the case in many applications discussed in this book.

1.3.1 The s-Transform

Let $f_X(x)$ be the PDF of the continuous random variable X that takes only nonnegative values; that is, $f_X(x) = 0$ for $x < 0$. The s-transform of $f_X(x)$, denoted by $M_X(s)$, is defined by:

$$M_X(s) = E\left[e^{-sX}\right] = \int_0^{\infty} e^{-sx} f_X(x)\,dx.$$

One important property of an s-transform is that when it is evaluated at the point $s = 0$, its value is equal to 1. That is,

$$M_X(s)\big|_{s=0} = \int_0^{\infty} f_X(x)\,dx = 1.$$

For example, the value of K for which the function $A(s) = K/(s+5)$ is a valid s-transform of a PDF is obtained by setting $A(0) = 1$, which gives:

$$K/5 = 1 \Rightarrow K = 5.$$

1.3.2 Moment-Generating Property of the s-Transform

One of the primary reasons for studying the transform methods is to use them to derive the moments of the different probability distributions. By definition:

$$M_X(s) = \int_0^{\infty} e^{-sx} f_X(x)\,dx.$$

Taking different derivatives of $M_X(s)$ and evaluating them at $s = 0$, we obtain the following results: