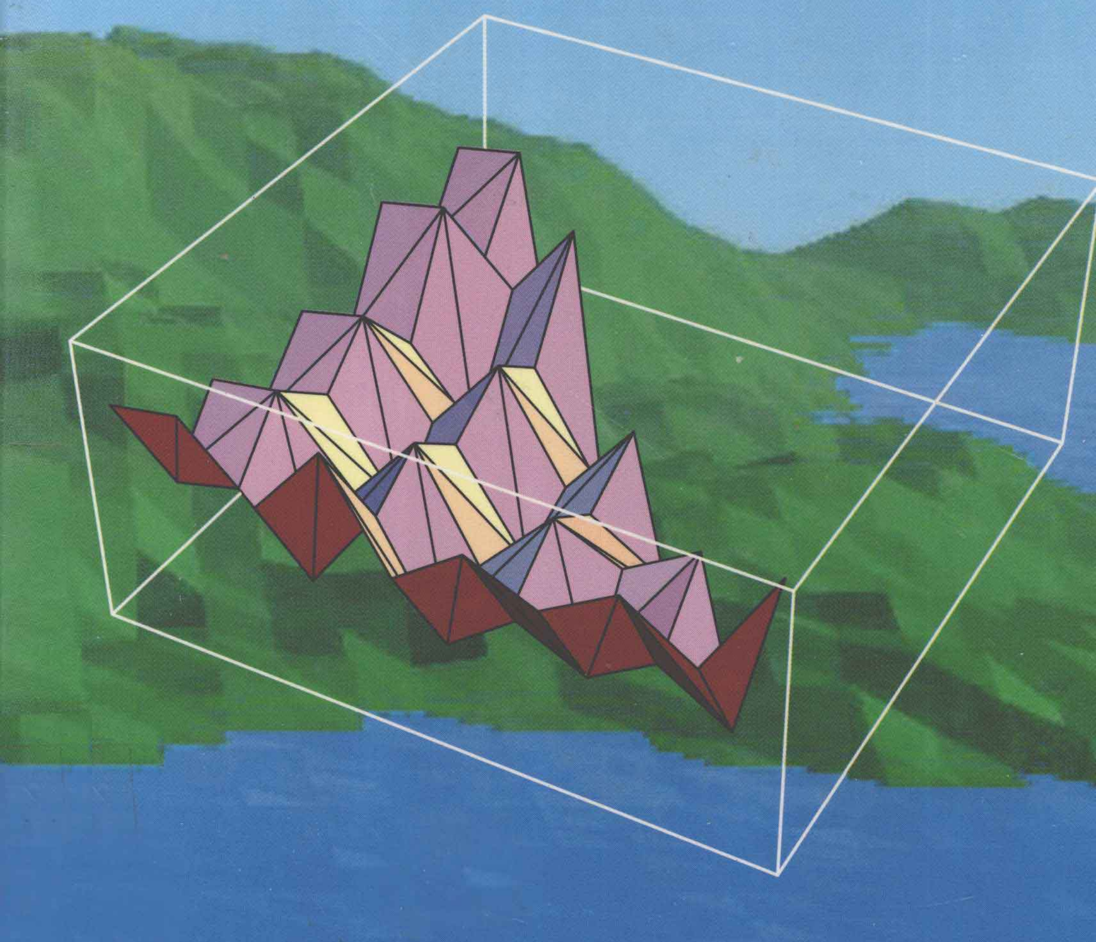
Fractal Functions, Fractal Surfaces, and Wavelets



Peter R. Massopust

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Peter R. Massopust
Sam Houston State University
Department of Mathematics
Huntsville, Texas

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Preface

This monograph gives an introduction to the theory of fractal functions and fractal surfaces with an application to wavelet theory. The study of fractal functions goes back to Weierstraß's nowhere differentiable function and beyond. However, it wasn't until the publication of B. Mandelbrot's book (cf. [123]) in which the concept of a fractal set was introduced and common characteristics of these sets were identified (such as nonintegral dimension and geometric self-similarity) that the theory of functions with fractal graphs developed into an area of its own. Seemingly different types of nowhere differentiable functions, such as those investigated by Besicovitch, Ursell, Knopp, and Kiesswetter, to only mention a few, were unified under the fractal point of view. This unification led to new mathematical methods and applications in areas that include: dimension theory, dynamical systems and chaotic dynamics, image analysis, and wavelet theory.

The objective of this monograph is to provide essential results from the theory of fractal functions and surfaces for those interested in this fascinating area, to present new and exciting applications, and to indicate which interesting directions the theory can be extended. The book is essentially self-contained and covers the basic theory and different types of fractal constructions as well as some specialized and advanced topics such as dimension calculations and function space theory.

The first part of the book contains background material and consists of four chapters. The first chapter introduces the relevant notation and terminology and gives a brief review of some of the basic concepts from classical analysis, abstract algebra and probability theory that are necessary for the remainder of the book. The reader who is not quite familiar with some of the material presented in this first chapter is referred to the bibliography where most of these concepts are defined and motivated. However, efforts were made to keep the mathematical requirements at a level where a graduate student with a solid background in the afore-mentioned areas will be able to work through most of the book.

The second chapter introduces same basic constructions of fractal sets. The first is based upon the approach by J. Hutchinson [98] and M. Barnsley and S. Demko [9] using what is now called an iterated function system. This method is then generalized and compared to M. Dekking's [45] construction of so-called recurrent sets associated with certain semigroup endomorphisms and C. Bandt's approach [5, 6] via topological Markov chains. Finally, a graph-directed fractal construction due to D. Mauldin and S. Williams [134] is presented. The emphasis is on iterated function systems and their generalizations; however. In this chapter the foundations for the rigorous treatment of univariate and multivariate fractal functions are laid.

Next, the concept of dimension of a set is introduced. This is done by first reviewing the different notions of dimension that are used to characterize and describe sets. The last two sections in this chapter are devoted to the presentation of dimension results for self-affine fractal sets.

A short chapter dealing with the fascinating theory of dynamical systems follows. The emphasis is on the geometric aspects of the theory and it is shown how they can be used to describe attractors of iterated function systems.

In the second part of this book, univariate and multivariate fractal functions are discussed. The fifth chapter introduces fractal functions as the fixed points of a Read-Bajraktarević operator. This approach differs from that undertaken by M. Barnsley [8] who introduced fractal functions for interpolation and approximation purposes. It is also shown how M. Dekking's approach to fractals can be used to define fractal functions and the iterative interpolation process of S. Dubuc and his co-workers is presented. The remainder of the chapter deals with different classes of fractal functions and discusses several of their properties.

Chapter 6 is devoted to dimension calculations. Formulae for the box dimension of the graphs of most of the fractal functions introduced in the previous chapter are presented here. The second part of the chapter deals with an interesting relationship between certain classes of smoothness spaces and the box dimension of the graphs of affine fractal functions.

In Chapter 7, the basic concepts and notions of wavelet theory are introduced, and it is demonstrated how a certain class of fractal functions generated by iterated function systems can be used to generate a multiresolution analysis of $L^2(\mathbb{R})$. This class of fractal functions then provides a new construction of continuous, compactly supported and orthogonal scaling functions and wavelets.

The next chapter introduces multivariate fractal functions. The graphs of these functions are called fractal surfaces. Properties of fractal surfaces are then discussed and formulae for the box dimension derived.

In order to construct multiresolution analyses based on the fractal surfaces defined in Chapter 8, the theory of Coxeter groups needs to be employed. This is done in Chapter 9, after some rudimentary concepts of this theory are introduced.

Because of the limited scope of this monograph, certain topics could not be covered. This includes a more in-depth presentation of the geometric theory of dynamical systems and the role fractals play in this theory. Furthermore, some of the work of T. Lindstrøm on nonstandard analysis, iterated function systems, fractals, and especially Brownian motion on fractals is beyond the limits of this book. The interesting work of J. Harrison dealing with geometric integration theory and fractals could also not be described. However, references pertaining to these as well as other topics are listed in the bibliography. The bibliography also contains research papers and books not explicitly used or mentioned in this monograph. They were included to give the reader a more well-rounded perspective of the subject.

This book grew out of the work of many mathematicians from several areas of mathematics, and the author has greatly benefited from numerous conversations and discussions with my colleagues. Special thanks go to Doug Hardin and Jeff Geronimo, who have influenced and shaped some of my thoughts and ideas. In particular, I am grateful to Doug Hardin for allowing me to use his *Mathematica* packages to make some of the figures in this monograph. I also wish to thank Patrick Van Fleet for introducing me to the theory of Dirichlet splines and special functions.

Working with Academic Press was a pleasure. I would like to especially express my gratitude to Christina Wipf, who gave me the idea of writing this monograph, and to Peter Renz, who guided me through the final stages.

Last but not least, I wish to thank my wife Maritza and my family for their continuous support and encouragement during the preparation of this monograph.

Peter R. Massopust

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Part I

Foundations

Chapter 1

Mathematical Preliminaries

This chapter provides most of the mathematical preliminaries necessary to understand the results in the following chapters. It is a mere collection of definitions and theorems given without a proof (the only exceptions are the Banach Fixed-Point Theorem, and the Existence Theorems for free semigroups and free groups). The bibliography contains a list of references in which all these results are motivated and proved. In a sense, this first chapter compiles notation and terminology and serves as a reference guide for the remainder of the book.

The relevant material is discussed in three sections: analysis and topology, probability theory, and algebra. The first section covers such basic topics as linear spaces, normed and metric spaces, point-set topology, measures, and the different notions of convergence encountered in analysis. In the second section, probability measures, distribution functions, random variables, and their interconnections are considered. Then the Lebesgue spaces are defined, the Riesz Representation Theorem is stated, and a brief overview of Markov processes and Markov chains is given. The last section deals with diagrams, semigroups, groups, and semigroup and group endomorphisms and introduces free semigroups and free groups. A brief review of category theory and direct and inverse limits is also provided.

1.1 Analysis and Topology

Throughout this monograph, $\mathbb{N} := \{1, 2, 3, \ldots\}$ denotes the set of natural numbers, $\mathbb{Z}$ the ring of integers, and $\mathbb{R}$ the field of real numbers. Let $\mathbb{K}$ be a subfield of $\mathbb{C}$, the field of complex numbers, and suppose that the mapping

$\alpha : \mathbb{C} \to \mathbb{C}$, $z = x + iy \mapsto \overline{z} = x - iy$ maps $\mathbb{K}$ into itself (α is called an *involuntary automorphism of* $\mathbb{C}$).

Suppose X and Y are linear spaces over $\mathbb{K}$.

***Definition* 1.1** 1. A mapping $\varphi : X \to \mathbb{K}$ is called semilinear or a semilinear form iff

   (a) $\forall x, x' \in X : \ \varphi(x + x') = \varphi(x) + \varphi(x')$.
   (b) $\forall x \in X \ \forall k \in \mathbb{K} : \ \varphi(kx) = k\varphi(x)$.

   If $\mathbb{K} \subseteq \mathbb{R}$ all semilinear forms are linear.

2. A mapping $\varphi : X \times X \to \mathbb{K}$ is called sesquilinear or a sesquilinear form iff

   (a) $\forall x, x', y \in X : \ \varphi(x + x', y) = \varphi(x, y) + \varphi(x', y)$.
   (b) $\forall k \in \mathbb{K} : \ \varphi(kx, y) = k\varphi(x, y)$.
   (c) $\forall x, y, y' \in X : \ \varphi(x, y + y') = \varphi(x, y) + \varphi(x, y')$.
   (d) $\forall k \in \mathbb{K} : \ \varphi(x, ky) = \overline{k}\varphi(x, y)$.

   If $\mathbb{K} \subseteq \mathbb{R}$, all sesquilinear forms are bilinear.

3. A sesquilinear form φ is called Hermitian iff $\forall x, y \in X : \ \varphi(x, y) = \overline{\varphi(y, x)}$ (if $\mathbb{K} \subseteq \mathbb{R}$ a Hermitian form is called symmetric).

4. A sesquilinear form φ is called positive definite, respectively positive semidefinite, iff $\forall x \in X, \ x \neq 0, \ \varphi(x, x) > 0$, respectively $\forall x \in X, \ \varphi(x, x) \geq 0$.

***Definition* 1.2** An inner product on a linear space X over $\mathbb{K}$ is a positive definite Hermitian sesquilinear form $\varphi : X \times X \to \mathbb{K}$.
The pair (X, φ) is called an inner product space. If X is a linear space over $\mathbb{R}$, respectively $\mathbb{C}$, (X, φ) is called an Euclidean space, respectively unitary space.

Notation. Instead of writing $\varphi(x, y)$, $x, y \in X$, the shorter notation $\langle x, y \rangle$ is sometimes used.

An inner product φ on X can be used to define the norm of an element $x \in X$, and the distance between two elements $x, y \in X$. More precisely, the *norm* of $x \in X$ is defined as

$$\|x\|_\varphi := \sqrt{\varphi(x, x)}, \tag{1.1}$$

and the *distance* between $x, y \in X$ by

$$d_\varphi := \|x - y\|_\varphi = \sqrt{\varphi(x - y, x - y)}. \tag{1.2}$$

Proposition 1.1 (Cauchy-Schwartz Inequality) *Let (X, φ) be an inner product space over $\mathbb{K}$. Then, for all $x, y \in X$,*

$$|\varphi(x, y)| \leq \sqrt{\varphi(x, x)}\sqrt{\varphi(y, y)} = \|x\|_\varphi \|y\||_\varphi, \tag{1.3}$$

with equality iff there exists a $k \in \mathbb{K}$ such that $x = ky$. ■

***Definition* 1.3** Suppose that X is a linear space over $\mathbb{K}$. A non-negative functional $\|\cdot\| : X \to \mathbb{R}$ is called a norm on X iff the following conditions hold:

- **(a)** $\forall x \in X : \ \|x\| \geq 0, \ \|0\| = 0.$
- **(b)** $\forall x \in X \ \forall k \in \mathbb{K} : \ \|kx\| = |k| \, \|x\|.$
- **(c)** $\forall x, y \in X : \ \|x + y\| \leq \|x\| + \|y\|.$
- **(d)** $\|x\| = 0 \Rightarrow x = 0.$

If only properties (a) — (c) are satisfied, $\|\cdot\|$ is called a semi-norm on X. The pair $(X, \|\cdot\|)$ is called a normed (linear) space.

Proposition 1.2 *Suppose (X, φ) is an inner product space over $\mathbb{K}$. Then $\|\cdot\|_\varphi : X \to \mathbb{K}$ as defined in (1.1) is a norm on X.* ■

***Definition* 1.4** Suppose M is a set. A mapping $d : M \times M \to \mathbb{R}$ is called a metric on M iff the following conditions are satisfied:

- **(a)** $\forall x, y \in M : \ d(x, y) \geq 0, \quad d(x, x) = 0.$
- **(b)** $\forall x, y \in M : \ d(x, y) = d(y, x).$
- **(c)** $\forall x, y, z \in M : \ d(x, z) \leq d(x, y) + d(y, z).$
- **(d)** $d(x, y) = 0 \Rightarrow x = y.$

If only properties (a) — (c) hold, then d is called a semi-metric on M.
The pair (X, d), where d is a (semi-)metric on the set X, is called a (semi-) metric space.

Proposition 1.3 *Suppose* $\|\cdot\|$ *is a norm, respectively a semi-norm, on a linear space* X *over* $\mathbb{K}$. *Then*

$$d(x,y) := \|x-y\|, \; x,y \in X, \tag{1.4}$$

defines a metric, respectively a semi-metric, on X. ■

A norm $\|\cdot\| : X \to \mathbb{R}$ on a linear space X over $\mathbb{K}$ induces in a canonical way a topology on X, the so-called *norm* or *strong topology.* At this point the definition of topology on a set M is recalled.

***Definition* 1.5** Let M be an arbitrary set and let $\mathcal{T}$ be a collection of subsets of M. Then $\mathcal{T}$ is called a topology on M provided

(a) For all i in some index set I, $T_i \in \mathcal{T} \Rightarrow \bigcup_{i\in I} T_i \in \mathcal{T}$.

(b) $T_1, \ldots, T_n \in \mathcal{T} \Rightarrow \bigcap_{i=1}^{n} T_i \in \mathcal{T}$.

(c) $M \in \mathcal{T}$, $\emptyset \in \mathcal{T}$.

The elements of $\mathcal{T}$ are called open sets and the pair $(X, \mathcal{T})$ a topological space.

The norm topology on X is then defined as follows: Let $A \subseteq X$, and let $B_r(a) := \{x \in X | \, \|x-a\| < r\}$ denote the ball of radius $r > 0$ centered at $a \in X$. The set A is called *open* iff for each $a \in A$ there exists a ball $B_r(a)$, $r > 0$, contained entirely in A. It is easy to show that $\mathcal{T}_{\|\cdot\|} := \{A \subseteq X | \, A$ is open$\}$ is a topology on X.

The topological space $(X, \mathcal{T}_{\|\cdot\|})$ is also *Hausdorff.*

***Definition* 1.6** A topology $\mathcal{T}$ on a set M is called Hausdorff iff two distinct points $x, y \in M$ can be separated by two disjoint sets U and V in $\mathcal{T}$, i.e., $\forall x,y \in M$, $x \neq y$, $\exists U, V \in \mathcal{T}$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

Suppose that X is a linear space over $\mathbb{K}$, and $\|\cdot\|_i : X \to \mathbb{R}$, $i = 1,2$, are arbitrary norms on X. $\|\cdot\|_1$ and $\|\cdot\|_2$ are called *equivalent*, written $\|\cdot\|_1 \approx \|\cdot\|_2$, iff there exist positive real numbers c_1 and c_2 such that for all $x \in X$,

$$\|x\|_1 \leq c_1 \, \|x\|_2, \quad \text{and} \quad \|x\|_2 \leq c_2 \, \|x\|_1. \tag{1.5}$$

Proposition 1.4 *1. All norms on* $\mathbb{R}^n$ *are equivalent.*

2. All norms on $\mathbb{R}^n$ generate the same topology. ■

***Definition* 1.7** Suppose $(X, \mathcal{T})$ is a topological space. $\mathcal{B} \subseteq \mathcal{T}$ is called a basis of $\mathcal{T}$ iff every open set is a union of elements of $\mathcal{B}$:

$$T \in \mathcal{T} \Rightarrow T = \bigcup_{B \in \mathcal{B}'} B,$$

with $\mathcal{B}' \subseteq \mathcal{B}$.

The following result gives necessary conditions for a subset $\mathcal{B}$ of $\mathcal{T}$ to be a base.

Proposition 1.5 *Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{B} \subseteq \mathcal{T}$. If $\mathcal{B}$ is a base of $\mathcal{T}$, then*

(a) $X = \bigcup_{B \in \mathcal{B}} B$;

(b) $\forall B_1, B_2 \in \mathcal{B}\ \forall x \in B_1 \cap B_2\ \exists B_x \in \mathcal{B}\colon x \in B_x \subseteq B_1 \cap B_2$. ■

The concept of topology allows one to precisely define notions such as *distance, convergence, and continuity.*

***Definition* 1.8** Let $\mathbf{D}$ be a non-empty set. A relation $\preceq$ on $\mathbf{D}$ is called directed iff it has the following properties:

(a) Reflexivity: $\forall \alpha \in \mathbf{D} :\ \alpha \preceq \alpha$.

(b) Transitivity: $\forall \alpha, \beta, \gamma \in \mathbf{D} :\ \alpha \preceq \beta,\ \beta \preceq \gamma \Rightarrow \alpha \preceq \gamma$.

(c) $\forall \alpha, \beta \in \mathbf{D}\ \exists \gamma \in \mathbf{D} :\ \alpha \preceq \gamma,\ \beta \preceq \gamma$.

A directed set is a set with a directed ordering.

Remark. Some authors define a directed set as a non-empty partially ordered set satisfying condition (c) above.

Let $(X, \mathcal{T})$ be a topological space. A *net* in X consists of a directed set $\mathbf{D}$ and a mapping $\delta : \mathbf{D} \to X$. It is common to write the image of $\alpha \in \mathbf{D}$ under δ in X as x_δ instead of $\delta(\alpha)$. Nets are then denoted by $\{x_\alpha\}_{\alpha \in \mathbf{D}}$, or simply by $\{x_\alpha\}$ if it is understood which directed set is meant. Clearly, every sequence in X is a net in X: take $\mathbf{D} = \mathbb{N}$ and $\preceq := \leq$.

Recall that a set $N \subseteq X$ is called a *neighborhood of* $x \in X$ iff N is a superset of an open set containing x.

***Definition* 1.9** Let $(X, \mathcal{T})$ be a topological space and let $\{x_\alpha\}$ be a net in X. A point $x \in X$ is called a limit point of $\{x_\alpha\}$ iff for any neighborhood N of x there exists an $\alpha_0 \in \mathbf{D}$ such that all x_α with $\alpha_0 \preceq \alpha$ are points in N.

Notation. If x is a limit point of a net $\{x_\alpha\}$, then one writes for short: $x_\alpha \to x$ (in $\mathcal{T}$).

The classical characterization of convergence (in the strong topology) is obtained by choosing X to be a normed linear space, $\mathcal{T} = \mathcal{T}_{\|\cdot\|}$, and $\mathbf{D} = \mathbb{N}$:

$$x_n \to x \iff \forall \varepsilon > 0\ \exists n_0\ \forall n \geq n_0 :\ \|x_n - x_0\| < \varepsilon.$$

Let $(X, \mathcal{T})$ and $(X', \mathcal{T}')$ be two topological spaces, and $F : X \to X'$ a mapping of sets. Then F is *continuous* iff $x_\alpha \to x$ in $(X, \mathcal{T})$ implies $F(x_\alpha) \to F(x)$ in $(X', \mathcal{T}')$, for every net $\{x_\alpha\}$ in X. A mapping $F : X \to X'$ is called a *homeomorphism* iff F is bijective and F and its inverse F^{-1} are continuous.

***Definition* 1.10** Let $(X, \mathcal{T})$ be a topological space and let $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ be the completed real line. A function $f : X \to \overline{\mathbb{R}}$ is called upper semi-continuous, respectively lower semi-continuous, at $x_0 \in X$ iff for all $\alpha \in \overline{\mathbb{R}}$ with $\alpha > f(x_0)$, respectively $\alpha < f(x_0)$, there exists a neighborhood N of x_0 in X such that for all $x \in N$ one has $\alpha > f(x)$, respectively $\alpha < f(x)$.
A function f is called upper semi-continuous, respectively lower semi-continuous, on X iff it is upper semi-continuous, respectively lower semi-continuous, at each $x_0 \in X$.

It is clear that if f is upper semi-continuous then $-f$ is lower semi-continuous. Also, if a function f is both upper and lower semi-continuous at $x_0 \in X$ then f is continuous at x_0. The next proposition characterizes lower semi-continuous functions.

Proposition 1.6 *A function f from a topological space $(X, \mathcal{T})$ into the completed real line $\overline{\mathbb{R}}$ is lower semi-continous iff for all $\alpha \in \overline{\mathbb{R}}$ the set*

$$f^{-1}(\alpha, \infty]$$

is open in X, or equivalently, the set

$$f^{-1}[-\infty, \alpha]$$

is closed in X. ■

***Definition* 1.11** Let $(X, \mathcal{T})$ be a topological space and let $\{x_\alpha\}$ be a net in X. A point $x \in X$ is called an accumulation point of $\{x_\alpha\}$ iff for every neighborhood N of x and any α_0 there exists an $\alpha \succeq \alpha_0$ such that $x_\alpha \in N$.