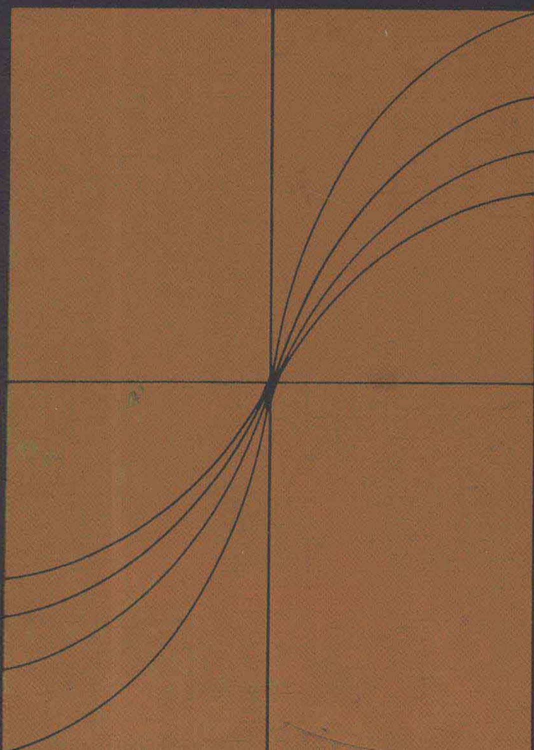
SECOND EDITION

RICHARD E. WILLIAMSON / HALE F. TROTTER

MULTIVARIABLE MATHEMATICS



Linear Algebra
Calculus
Differential Equations

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Differential Equations

SECOND EDITION

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MULTIVARIABLE MATHEMATICS

Preface

This book covers the material that is usually taken after the first course in calculus: linear algebra, multivariable calculus, and differential equations. We have tried to make the book flexible so it is not necessary for the instructor to cover the topics in the precise order that they appear. Note also that the sections in each chapter are divided into subsections in order of decreasing importance so that some sections can be omitted. Some possible course outlines are given below; they are designed for a one-term introductory course of about forty class meetings of fifty minutes duration. In each case, a selection of the remaining material will do for a second term. Some subsections may be omitted at the option of the instructor.

Basic Course:

Ch. 1, §1-5; Ch. 2, §1-4; Ch. 3, §1-4;
Ch. 5, §1; Ch. 8, §1-7; Ch. 9, §1-5;
Ch. 10, §1, 2, 4; Ch. 13, §1-3; Ch. 14, §1-3.

Emphasis on Linear Algebra:

Ch. 1, §1-5; Ch. 2, §1-4; Ch. 3, §1-5;
Ch. 4, §1-3; Ch. 5, §1; Ch. 8, §1-7; Ch. 9, §1-4;
Ch. 10, §1, 2; Ch. 13, §1-3.

Emphasis on Differential Equations:

Ch. 1, §1-5; Ch. 2, §1-4; Ch. 8, §1-7; Ch. 10, §1-2;
Ch. 13, §1-3; Ch. 14, §1-3; Ch. 15, §1-3; Ch. 16, §1-3.

Our aim has been to provide standard problem-solving techniques and to show how the framework of linear algebra displays the various topics to their best advantage. In particular, it is not necessary to study the formal proofs that we provide, although these may often be illuminating. The definitions and statements of theorems are included to show how the subject matter can be organized around a few central ideas. The examples are really the heart of the book; to complement them, we have included many routine, computational exercises.

This edition has been improved by many thoughtful suggestions from other mathematicians. We would like to thank Professor John D. Baum of Oberlin College, Professor Charles D. Brown of the University of Tennessee at Chattanooga, and Professor Lisl Gaal of the University of Minnesota. We would particularly like to mention the contributions of Professor Robert Messer of Vanderbilt University and Professor Ann Stehney of Wellesley College. The problem solutions were prepared by Gregory Call. Nancy French and Helen Hanchett did most of the typing.

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MULTIVARIABLE MATHEMATICS

Contents

One

Vectors

Vectors originated as geometric objects, with magnitude and direction, suitable for representing physical quantities such as displacements, velocities, and forces. It turned out later that a variety of topics in pure and applied mathematics could be unified and simplified by introducing a more general algebraic concept of vector.

This first chapter introduces vectors in algebraic terms, but is chiefly concerned with their geometric interpretation. The chapter is fundamental for the rest of the book, because the possibility of visualizing problems geometrically is one of the major advantages of using vectors.

Section 1 / COORDINATE VECTORS

We use the script letter $\mathcal{R}$ to stand for the set of all real numbers, $\mathcal{R}^2$ for the set of ordered pairs (x_1, x_2), $\mathcal{R}^3$ for the set of ordered triples (x_1, x_2, x_3), and in general $\mathcal{R}^n$ for the set of n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers. Thus a statement about $\mathcal{R}^n$ is a statement about all the sets $\mathcal{R}^2$, $\mathcal{R}^3$, $\mathcal{R}^4$, etc., at once. $\mathcal{R}^1$ means the same thing as $\mathcal{R}$. Boldface letters $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, etc., will stand for n-tuples, whereas ordinary lightface letters will stand for single real numbers. In particular, we may write $\mathbf{x} = (x, y)$ or $\mathbf{x} = (x, y, z)$ for general pairs and triples to save writing subscripts.

For any real number r and n-tuple $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, we define the **numerical multiple** $r\mathbf{x}$ to be the n-tuple

$$(rx_1, rx_2, \ldots, rx_n),$$

obtained by multiplying each entry x_k by r.

Example 1. If we take $\mathbf{x} = (1, 2)$ in $\mathcal{R}^2$ and $r = 3$, then

$$r\mathbf{x} = 3(1, 2) = (3, 6).$$

Similarly, with $\mathbf{x} = (1, 2, -3)$ in $\mathcal{R}^3$ and $r = -2$,

$$r\mathbf{x} = -2(1, 2, -3) = (-2, -4, 6).$$

For any two n-tuples $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ in $\mathcal{R}^n$, we define the **sum** $\mathbf{x} + \mathbf{y}$ to be the n-tuple

$$(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n),$$

obtained by adding corresponding entries x_k and y_k.

Example 2. With $\mathbf{x} = (1, 2)$ and $\mathbf{y} = (2, -3)$ both in $\mathcal{R}^2$,

$$\mathbf{x} + \mathbf{y} = (1, 2) + (2, -3) = (3, -1).$$

In $\mathcal{R}^3$, with $\mathbf{x} = (0, 2, 4)$ and $\mathbf{y} = (-1, -2, 2)$, we have

$$\mathbf{x} + \mathbf{y} = (0, 2, 4) + (-1, -2, 2) = (-1, 0, 6).$$

The two vector operations, numerical multiplication and vector addition, are natural extensions of the addition and multiplication of single real numbers; one of their advantages is that they provide a simplified notation for carrying out several operations at once.

Example 3. Let $\mathbf{x} = (2, -1, 0)$ and $y = (0, -1, -2)$. Then

$$\begin{aligned} 2\mathbf{x} + \mathbf{y} &= 2(2, -1, 0) + (0, -1, -2) \\ &= (4, -2, 0) + (0, -1, -2) = (4, -3, -2). \end{aligned}$$

Similarly,

$$\begin{aligned} 3\mathbf{x} - 2\mathbf{y} &= 3(2, -1, 0) - 2(0, -1, -2) \\ &= (6, -3, 0) + (0, 2, 4) = (6, -1, 4). \end{aligned}$$

We write $-\mathbf{x}$ for the numerical multiple $(-1)\mathbf{x}$, and $\mathbf{x} - \mathbf{y}$ as an abbreviation for $\mathbf{x} + (-\mathbf{y})$. We use 0 to denote an n-tuple consisting entirely of zeros. The zero notation is ambiguous since, for example, 0 may stand for (0, 0) in one formula and for (0, 0, 0) in another. The ambiguity seldom causes any confusion since in most contexts only one interpretation makes sense. For instance, if $\mathbf{z} = (-2, 0, 3)$, then in the formula $\mathbf{z} + 0$, the 0 must stand for (0, 0, 0) since addition is defined only between n-tuples with the same number of entries.

The following formulas hold for arbitrary $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ in $\mathcal{R}^n$ and arbitrary numbers r, s. They state rules for our new operations of addition and numerical multiplication very closely analogous to the familiar distributive, commutative, and associative laws for ordinary addition and multiplication of numbers.

1. $r\mathbf{x} + s\mathbf{x} = (r + s)\mathbf{x}$.
2. $r\mathbf{x} + r\mathbf{y} = r(\mathbf{x} + \mathbf{y})$.
3. $r(s\mathbf{x}) = (rs)\mathbf{x}$.
4. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.
5. $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$.
6. $\mathbf{x} + \mathbf{0} = \mathbf{x}$.
7. $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$.
8. $1\mathbf{x} = \mathbf{x}$.

These rules are straightforward consequences of the definitions of the vector operations and of the laws of arithmetic. For illustration, we give a formal proof of formula 2.

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$, and let r be a real number. Then

$$r\mathbf{x} = (rx_1, rx_2, \ldots, rx_n) \qquad \text{[definition of numerical multiplication]}$$

$$r\mathbf{y} = (ry_1, ry_2, \ldots, ry_n) \qquad \text{[definition of numerical multiplication]}$$

and so

$$r\mathbf{x} + r\mathbf{y} = (rx_1 + ry_1, rx_2 + ry_2, \ldots, rx_n + ry_n) \qquad \text{[definition of addition].}$$

On the other hand,

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \qquad \text{[definition of addition]}$$

and so

$$r(\mathbf{x} + \mathbf{y}) = (r(x_1 + y_1), r(x_2 + y_2), \ldots, r(x_n + y_n)) \qquad \text{[definition of numerical multiplication].}$$

By the distributive law of ordinary arithmetic, $r(x_1 + y_1) = rx_1 + ry_1$, $r(x_2 + y_2) = rx_2 + ry_2$, etc., and therefore the n-tuples $r\mathbf{x} + r\mathbf{y}$ and $r(\mathbf{x} + \mathbf{y})$ are the same, as was to be proved.

Any set with operations of addition and multiplication by real numbers defined in such a way that the rules 1 through 8 hold is called a **vector space,** and its elements are called **vectors.** We shall discuss some other vector spaces in Chapter 3, but for the present "vector" may be taken to mean "element of $\mathcal{R}^n$" for some n. Numbers are sometimes called **scalars** when emphasis on the distinction between numbers and vectors is wanted. In physics, for example, mass and energy may be referred to as scalar quantities, in distinction to vector quantities such as velocity or momentum. The term *scalar multiple* is synonymous with what we have called numerical multiple.

The special vectors

$$\mathbf{e}_1 = (1, 0), \quad \mathbf{e}_2 = (0, 1) \qquad \text{in } \mathcal{R}^2,$$

$$\mathbf{e}_1 = (1, 0, 0), \quad \mathbf{e}_2 = (0, 1, 0), \quad \mathbf{e}_3 = (0, 0, 1) \qquad \text{in } \mathcal{R}^3,$$

and in general

$$\mathbf{e}_1 = (1, 0, \dots, 0), \quad \mathbf{e}_2 = (0, 1, \dots, 0), \quad \dots, \quad \mathbf{e}_n = (0, 0, \dots, 1) \qquad \text{in } \mathcal{R}^n$$

have the property that, if $\mathbf{x} = (x_1, \dots, x_n)$ is an arbitrary vector, then

$$\mathbf{x} = x_1\mathbf{e}_1 + \dots + x_n\mathbf{e}_n.$$

Example 4. Given any vector $\mathbf{x} = (x_1, x_2)$ in $\mathcal{R}^2$, we have

$$(x_1, x_2) = x_1(1, 0) + x_2(0, 1)$$
$$= x_1\mathbf{e}_1 + x_2\mathbf{e}_2;$$

in particular,

$$(2, -3) = 2\mathbf{e}_1, - 3\mathbf{e}_2.$$

In $\mathcal{R}^3$ we have

$$(x_1, x_2, x_3) = x_1(1, 0, 0) + x_2(0, 1, 0) + x_3(0, 0, 1)$$
$$= x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3;$$

in particular,

$$(1, 2, -7) = \mathbf{e}_1 + 2\mathbf{e}_2 - 7\mathbf{e}_3.$$

Notational warning. The notation $\mathbf{e}_k$ means different things depending on whether it stands for a vector in $\mathcal{R}^2$, or $\mathcal{R}^3$, or $\mathcal{R}^n$ for some other n. In this book it will always be clear from the context how many entries a vector $\mathbf{e}_k$ has.

Because every element of $\mathcal{R}^n$ can be so simply represented using the vectors $\mathbf{e}_k$, the set of vectors $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is called the **natural basis** for $\mathcal{R}^n$. The entries in

$$\mathbf{x} = (x_1, \dots, x_n)$$

are then called the **coordinates** of $\mathbf{x}$ relative to the natural basis.

A sum of numerical multiples $a_1\mathbf{e}_1 + \dots + a_n\mathbf{e}_n$ is called a linear combination of the vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$. More generally, a sum of multiples $a_1\mathbf{x}_1 + \dots + a_n\mathbf{x}_n$ is called a **linear combination** of the vectors $\mathbf{x}_1, \dots, \mathbf{x}_n$.

Example 5. The equation

$$(2, 3) = 2\mathbf{e}_1 + 3\mathbf{e}_2$$

shows (2, 3) written as a linear combination of $\mathbf{e}_1$ and $\mathbf{e}_2$ in $\mathcal{R}^2$. But the vector (2, 3) can also be written as a linear combination of (1, 1) and (1, −1) as follows:

$$(2, 3) = \tfrac{5}{2}(1, 1) - \tfrac{1}{2}(1, -1).$$

In $\mathcal{R}^3$, the equation

$$(2, 3, 4) = 4(1, 1, 1) - 1(1, 1, 0) - 1(1, 0, 0)$$

shows the vector (2, 3, 4) represented as a linear combination of the vectors (1, 1, 1), (1, 1, 0,), and (1, 0, 0).

Example 6. To express (1, 2) as a linear combination of (1, 1) and (3, 4) we must find numbers x and y such that

$$(1, 3) = x(1, 1) + y(3, 4),$$

or equivalently,

$$(1, 3) = (x + 3y, x + 4y).$$

Thus we need to solve

$$x + 3y = 1$$
$$x + 4y = 3$$

for x and y. Subtracting the first equation from the second gives $y = 2$. Then setting $y = 2$ in the first equation gives $x = -5$. Hence

$$(1, 3) = -5(1, 1) + 2(3, 4)$$

expresses (1, 3) as a linear combination of (1, 1) and (3, 4).

As to why it is useful to consider vectors in $\mathcal{R}^n$ for arbitrarily large values of n, consider the following.

Example 7. A realistic model for the economic growth (or decline) of a country needs to take into account the production and consumption of thousands of commodities. Thus a production vector $\mathbf{p}$ in $\mathcal{R}^n$ for $n = 5000$ might represent the amounts of each commodity produced by a country in a single year, with a corresponding consumption vector $\mathbf{c}$ representing the consumption of those commodities, listed in the same order. The difference vector $\mathbf{p} - \mathbf{c}$ would represent the excess of production over consumption for the same period. The number $n = 5000$ was chosen arbitrarily and is probably too small for any but a very simple economy, but electronic computers are capable of handling much larger vectors, so it is evident that while most of our examples will have to do with $n = 2$ or 3, the possibility of encountering much larger values of n should be kept open.

EXERCISES

1. Let $\mathbf{x} = (-3, 4)$ and $\mathbf{y} = (2, 2)$. Compute

(a) $\mathbf{x} + \mathbf{y}$. (b) $\mathbf{x} + 2\mathbf{y}$.
(c) $2\mathbf{x} + 3\mathbf{y}$. (d) $-\mathbf{x} + \mathbf{y} - (1, 4)$.

2. Let $\mathbf{x} = (3, -1, 0)$, $\mathbf{y} = (0, 1, 5)$, and $\mathbf{z} = (2, 5, -1)$. Compute

(a) $3\mathbf{x}$. (b) $\mathbf{y} + \mathbf{z}$.
(c) $4\mathbf{x} - 2\mathbf{y} + 3\mathbf{z}$. (d) $-\mathbf{y} + (1, 2, 1)$.

3. Find numbers a and b such that $a\mathbf{x} + b\mathbf{y} = (9, -1, 10)$, where $\mathbf{x}$ and $\mathbf{y}$ are as in Exercise 2. Is there more than one solution?

4. Show that no choice of numbers a and b can make $a\mathbf{x} + b\mathbf{y} = (3, 0, 0)$, where $\mathbf{x}$ and $\mathbf{y}$ are as in Exercise 2. For what value(s) of c (if any) can the equation $a\mathbf{x} + b\mathbf{y} = (3, 0, c)$ be satisfied?

5. Write out proofs for (a) rule 3 and (b) rule 4 on page 3, giving precise justification for each step.

6. Prove that the representation of a vector $\mathbf{x}$ in $\mathcal{R}^n$ in terms of the natural basis is unique. That is, show that if

$$x_1\mathbf{e}_1 + \ldots + x_n\mathbf{e}_n = y_1\mathbf{e}_1 + \ldots + y_n\mathbf{e}_n,$$

then $x_k = y_k$ for $k = 1, \ldots, n$. (*Hint*. Suppose that $x_k \neq y_k$ for some k.)

7. Represent the first vector as a linear combination of the remaining vectors, either by inspection or by solving an appropriate system of equations.

(a) $(2, -7); (1, 1), (1, -1)$.
(b) $(-2, 3); \mathbf{e}_1, \mathbf{e}_2$.
(c) $(2, 3, 4); (1, 1, 1), (1, 2, 1), (-1, 1, 2)$.

8. Let $\mathbf{x} = (5, 500, 10)$ represent the amount of ink, paper, and binding material needed to produce a single copy of some book and let $\mathbf{y} = (4, 800, 90)$ be the same vector for some other book. Interpret $100\mathbf{x} + 50\mathbf{y}$. What is an interpretation for $100\mathbf{x} - 50\mathbf{y}$?

9. Use rules 1 through 8 on page 3 to simplify the following.

(a) $2(3\mathbf{x} - 2\mathbf{y} + \mathbf{z}) - 4\mathbf{x}$.
(b) $\frac{1}{2}(\mathbf{x} + \mathbf{y}) - \mathbf{y}$.
(c) $\mathbf{x} + (\mathbf{x} + (\mathbf{x} + \mathbf{y}))$.
(d) $2\mathbf{x} + 3\mathbf{y} + 3\mathbf{x} - \mathbf{z}$.

10. A small factory produces four different products whose corresponding wholesale prices in dollars are given by the vector $\mathbf{w} = (50, 75, 100, 190)$. The retail price vector is $\mathbf{r} = (100, 150, 200, 300)$. The daily production vector in numbers of each product is $\mathbf{p} = (25, 25, 15, 10)$.

(a) What is the retailer's profit vector for the four products?
(b) If the wholesale price vector is doubled, what happens to the retailer's profit vector?
(c) What happens to the retailer's profit vector if the retail prices are each increased by 10% and the wholesale prices are left unchanged?
(d) What is the total production vector for the production of a five-day week?

11. The temperatures at 50 sites in a building are monitored by a remote computer. Suppose that $x_k(t)$ is the temperature at the kth site at time t as measured on a 24-hour clock. Then the vector $\mathbf{x}(t) = (x_1(t), \ldots, x_{50}(t))$ represents the temperatures in the entire building at time t. Write an expression in terms of $\mathbf{x}$ for the average temperature vector using the readings at $t = 8, 12, 16$, and 21.

12. Express each of the following vectors as a linear combination of $\mathbf{e}_1$ and $\mathbf{e}_2$ in $\mathcal{R}^2$, or of $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ in $\mathcal{R}^3$.

(a) $2(1, 2) - 3(-1, 4)$.
(b) $(1, 4) - (2c, d)$.
(c) $(1, 0, 1) + 3(2, 3, -1)$.
(d) $(x, y, z) + (z, y, x)$.