CAMBRIDGE TRACTS IN MATHEMATICS

138

RANDOM WALKS ON INFINITE GRAPHS AND GROUPS

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CAMBRIDGE UNIVERSITY PRESS

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UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521552929

First published 2000
This digitally printed version (with corrections) 2008

A catalogue record for this publication is available from the British Library

ISBN 978-0-521-55292-9 hardback
ISBN 978-0-521-06172-8 paperback

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General Editors

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PREFACE

"Random walks" is a topic situated somewhere in between probability, potential theory, harmonic analysis, geometry, graph theory, and algebra. The beauty of the subject stems from this linkage, both in the way of thinking and in the methods employed, of different fields.

Let me briefly declare what - in my viewpoint - random walks are. These are time-homogeneous Markov chains whose transition probabilities are in some way (to be specified more precisely in each case) adapted to a given structure of the underlying state space. This structure may be geometric or algebraic; here it will be discrete and infinite. Typically, we shall use locally finite graphs to view the structure. This also includes groups via their Cayley graphs. From the probabilistic viewpoint, the question is what impact the particular type of structure has on various aspects of the behaviour of the random walk, such as transience/recurrence, decay and asymptotic behaviour of transition probabilities, rate of escape, convergence to a boundary at infinity and harmonic functions. Vice versa, random walks may also be seen as a nice tool for classifying, or at least describing the structure of graphs, groups and related objects.

Of course, random walks on *finite* graphs and groups are a fascinating topic as well, and have had an enormous renaissance in the last decade: a book written by two major experts, D. Aldous and J. Fill, is about to appear.

Some might object that any countable Markov chain may be viewed on a directed graph, so that our notion of random walks coincides with arbitrary Markov chains. However, our point of view is reversed: what we have in mind is to start with a graph, group, etc., and investigate the interplay between the behaviour of random walks on these objects on one hand and properties of the underlying structure itself on the other.

Historically, I believe that this spirit of approaching the theory of random walks on infinite graphs has its roots in the 1921 paper by Pólya [269], whose nice title - translated into English - is "On an exercise in probability concerning the random walk in the road network". There, Pólya shows that simple random walk in the two-dimensional Euclidean grid is recurrent, while it is transient in higher dimensions. This change of behaviour between plane and space provided inspiration for much further work. However, it took 38 years until what I (personal opinion !) consider the next "milestones". In 1959, Nash-Williams published his paper "Random walks and electric currents in networks" [245], the first to link recurrence and structural properties of networks (i.e., reversible Markov chains). This paper -

not written in the style of the mainstream of mathematics at that time - remained more or less forgotten until the 80s, when it was rediscovered by T. Lyons, Doyle and Snell, Gerl, and others. The second 1959 milestone was Kesten's "Symmetric random walks on groups" [198], founding the theory of random walks on (infinite) groups and also opening the door from random walks to amenability and other topics of harmonic and spectral analysis.

Another direct line of extension of Pólya's result is to consider sums of i.i.d. random variables taking their values in $\mathbb{Z}^d$ - this was done to perfection in Spitzer's beautiful "Principles of Random walk" [307] (first edition in 1964), which is still the most authoritative and elegant source available. Spitzer's book also contains a considerable amount of potential theory. Note that Markov chains and discrete potential theory were born more or less simultaneously (while classical potential theory had already been very well developed before its connection with Brownian motion was revealed, and one still encounters analysts who deeply mistrust the so-called probabilistic proofs of results in potential theory - probably they believe that the proofs themselves hold only almost surely). Although not being directly concerned with the type of structural considerations that are inherent to random walks, I consider the third 1959 milestone to be Doob's "Discrete potential theory and boundaries" [101]. In the sixties, potential and boundary theory of denumerable Markov chains had a strong impetus promoted by Doob, Hunt, Kemeny, Snell, Knapp and others, before being somewhat "buried" under the burden of abstract potential theory. Doob's article immediately led to considerations in the same spirit that we have in mind here, the next milestone being the note of 1961 by Dynkin and Malyutov [111]. This contains the first structural description of the Martin boundary of a class of random walks and is also - together with Kesten [198] - the first paper where one finds the principal ingredients for computations regarding nearest neighbour random walks on free groups and homogeneous trees. Indeed, it is amusing to see how many people have been redoing these computations for trees in the belief of being the first to do so.

It was in a paper on boundaries that Kesten [201] indicated a problem which then became known as "Kesten's conjecture": classify those (finitely generated) groups which carry a recurrent random walk, the conjecture (not stated explicitly by Kesten) being that such a group must grow polynomially with degree at most two. It is noteworthy that the analogous problem was first settled in the 70s for connected Lie groups, see Baldi [17]. The Lie case is not easier, but there were more analytical and structural tools available at the time. The solution in the discrete case became possible by Gromov's celebrated classification of groups with polynomial growth [149] and was carried out in a remarkable series of papers by Varopoulos, who gave the final answer in [325]. In the 80s, random walks on graphs have been repopularized, owing much to the beautiful little book by Doyle and Snell [103]. However, this discussion of selected "milestones" is bringing me too close to the present, with many of the actors still on stage and the future to judge. Other important work from the late 50s and the 60s should also be mentioned here, such as that of Choquet and Deny [74] and - in particular - Furstenberg [124].

Let me return from this "historical" excursion. This book grew out of a long survey paper that I published in 1994 [348]. It is organized in a similar way, although here, less material is covered in more detail.

Each of the four chapters is built around one specific type of question concerning the behaviour of random walks, and answers to this question are then presented for various different structures, such as integer lattices, trees, free groups, plane tilings, Gromov-hyperbolic graphs, and so on. At the beginning, I briefly considered using the "orthogonal" approach, namely to order by types of structures, for example, saying first "everything" about random walks on integer lattices, then nilpotent groups and graphs with polynomial growth, trees, hyperbolic graphs, and so on. Some thought convinced me that this was not feasible. Thus, the same classes of structures will be encountered several times in this book. For example, the reader who is interested in results concerning random walks and trees will find these in paragraphs/sections 1.D, 5, 6.B, 10.C, 12.C, 19, 21.A and 26.A, tilings and circle packings are considered in 6.C-D, 10.C and 23, and the integer grids and their generalizations appear in 1.A, 6.A, 8.B, 13 and 25. Regarding the latter, I obviously did not aim at an exposition as complete as that of Spitzer had been in its time. Most likely, every reader will find a favorite among the topics in random walk theory that are not covered here (such as random walks on direct limits of finite groups, ratio limit theorems, or random walks in random environment).

A short word on notation. Instead of using further exotic alphabets, I decided not to reserve a different symbol for each different object. For example, the symbol Φ has different meanings in Sections 6, 9 and 12, and this should be clear from the context.

I started writing this book at the beginning of 1995 (one chapter per year). Thus, Chapter I is the oldest one among the material presented here, and so on. I decided not to make a complete updating of this material to the state of the art of today (1999) - otherwise I could never stop writing. In particular, the 90s saw the emergence of a new, very strong group of random walkers (and beyond) in Israel and the US (I. Benjamini, R. Lyons, Y. Peres, O. Schramm, ...) whose work is somewhat underrepresented here by this reason. On the other hand (serving as an excuse for me), two of them (Lyons and Peres) are currently writing their own book on "Probability on Trees and Networks" that can be expected to be quite exciting.

Many mathematical monographs of today start with two claims. One is to be self-contained. This book is *not* self-contained by the nature of its topic. The other claim is to be usable for graduate students. It has been my experience that usually, this must be taken with caution and is mostly true only in the presence of a guiding hand that is acquainted with the topic. I think that this is true here as well. Proofs are sometimes a bit condensed, and it may be that even readers above the student level will need pen and paper when they want to work through them seriously - in particular because of the variety of different methods and techniques that I have tried to unite in this text. This does not mean that parts of this book could not be used for graduate or even undergraduate courses. Indeed, I have taught parts of this material on several occasions, and at various levels.

Anyone who has written a book will have experienced the mysterious fact that a text of finite length may contain an infinity of misprints and mistakes, which apparently were not there during your careful proof-reading. In this sense, I beg excuse for all those flaws whose mysterious future appearance is certain.

In conclusion, let me say that I have learned a lot in working on this book, and also had fun, and I hope that this fun will "infect" some of the readers too.

Milano, July 1999 W.W.

ERRATA

The preparation of the paperback edition gave me the opportunity to prepare these 3 pages containing the corrections of a few misprints (many more will have remained) and two "true" mistakes, as well as a missing reference. Graz (Austria), September 2007, Wolfgang Woess

Page 4, lines 13–14 from top. If $n \to \infty$ then $q_n/n \to 1/m$ [in the proof of Lemma 1.9]

Page 63, line 7 from bottom. Reference to (6.10) [instead of (6.9)].

Page 82, lines 12–15 from top. In (2), (3) and (4): $\rho(P) < 1$ [instead of $\rho(P) > 1$].

Page 167, line 4 from bottom. for all $r, n \in \mathbb{N}$ [instead of $m, n \in \mathbb{N}$].

Page 170, proof of Theorem 15.15. The mistake is that the measure μ on line 7 is not symmetric. The proof should start as follows.

Let μ_0 and ν_0 be the equidistributions on $\{\mathbf{0}, \pm\mathbf{e}_i : i = 1, \dots, d\} \subset \mathbb{Z}^d$ and on $\mathfrak{A}$, respectively. Via the embedding of $\mathbb{Z}^d$ and $\mathfrak{A}$ into $\mathbb{Z}^d \wr \mathfrak{A}$, both are also considered as measures on the wreath product. For the proof, in view of Corollary 15.5, it is sufficient to consider the random walk on $\mathbb{Z}^d \wr \mathfrak{A}$ whose law is $\mu = \nu_0 * \mu_0 * \nu_0$ that is,

$$\mu(y, \eta) = \begin{cases} \mu_0(y)/|\mathfrak{A}|^2, & \text{if } \eta \in \{\eta_a + T_y \eta_b : a, b \in \mathfrak{A}\}, \\ 0, & \text{otherwise.} \end{cases}$$

Since $\nu_0 * \nu_0 = \nu_0$, we have $\mu^{(n)} = (\nu_0 * \mu_0)^{(n)} * \nu_0$. Consider i.i.d. random variables (K_n, V_n), where $K_n \in \mathbb{Z}^d$ has distribution μ_0 and the $\mathfrak{B}$-valued random variables V_n are all equidistributed on the set of configurations $\eta \in \mathfrak{B}$ with $\operatorname{supp} \eta \subset \{\mathbf{0}\}$, and K_n and V_n are independent. Then $\mu^{(n)}$ is the distribution of

$$\left(S_n, \sum_{j=1}^{n+1} T_{S_{j-1}} V_j \right) \in \mathbb{Z}^d \wr \mathfrak{A},$$

where $S_n = K_1 + \dots + K_n$ is the random walk on $\mathbb{Z}^d$ with law μ_0, with $S_0 = \mathbf{0}$.

The proof of the lower bound is then precisely as on page 170, taking into account that on line 10 from bottom, the middle term of the inequality has to be $\mathbb{P}_0[\max\{|S_j| : j \le n\} \le r]^2/|A_r|$ [the square was missing].

For the upper bound, the summation over j on page 171, lines 2–4 from top, should go up to $n+1$ instead of n, so that line 4 becomes

$$= \mathbb{E}_0\big(|\mathfrak{A}|^{-|D_n|} \mid S_n = 0\big)\,\mathbb{P}_0[S_n = 0] = \mathbb{E}_0\big(|\mathfrak{A}|^{-|D_n|}\,\mathbf{1}_{[S_n=0]}\big)\,,$$

after which the proof concludes as before.

Page 185, line 6 from bottom. The last term of the sum is $\cdots + C(x|\mathrm{r})\sqrt{\mathrm{r} - z/\xi_\ell}^{\,5}$ [coefficient $C(x|\mathrm{r})$ instead of $C(x|\mathrm{r}/\xi_\ell)$].

Page 215, line 5 from top. $p^{(n)}(x,y) \sim A\Big(1 + \dfrac{q-1}{q+1}d(x,y)\Big)\cdots$ [asymptotic equivalence instead of equality].

Page 294–295, Proof of Theorem 27.1. As pointed out by the late Martine Babillot, there is a mistake in the proof on page 295, lines 8–9: it does not follow from the preceding arguments that $\frac{1}{1-c_{n-1}}(h_1 - c_{n-1}\cdot h_2) \in \mathcal{C}_\xi$. We explain how the proof can be repaired by re-ordering the material.

The initial piece remains the same until the displayed formula on page 194, lines 4–3 from bottom, which contains some misprints. The material starting with this formula and ending on page 295, line 11 should be replaced by the following:

$$\begin{aligned}\frac{K(x,y_n)}{K(x,y_n')} &= \frac{F(x,y_n)F(o,y_n')}{F(o,y_n)F(x,y_n')}\\ &\ge \frac{F(x,v)F(v,y_n)\,F(o,v)F(v,y_n')}{C(2\delta)F(o,v)F(v,y_n)\,C(2\delta)F(x,v)F(v,y_n')} = \frac{1}{C(2\delta)^2}\,.\end{aligned}$$

Having proved (27.15), we now let L_ξ be the set of all limit points in the Martin boundary $\mathcal{M}(P)$ of sequences in X which converge to ξ in the hyperbolic topology. Bounded range implies that $K(\cdot,\alpha) \in \mathcal{H}^+(P)$ for every $\alpha \in L_\xi$. By (27.15), $K(\cdot,\alpha) \ge \varepsilon_1\, K(\cdot,\beta)$ for all $\alpha,\beta \in L_\xi$.

We next show in Step 2 that there is $\alpha \in L_\xi$ such that $K(\cdot,\alpha)$ is minimal harmonic. Then the last inequality will imply that $K(\cdot,\beta) = K(\cdot,\alpha)$ for all $\beta \in L_\xi$, that is, L_ξ consists of the single point α. The latter is then the natural image of ξ, completing Step 1.

Step 2. Let $\pi(o,\xi)$ be a geodesic from o to ξ. There must be a sequence (x_n) of points on $\pi(o,\xi)$ such that $|x_{n+1}| > |x_n|$ and $x_n \to \alpha \in L_\xi$ in the Martin topology. We define $\mathcal{H}_\alpha = \{h \in \mathcal{H}^+ : \sup_x h(x)/K(x,\alpha) = 1\}$. If we can show that $\mathcal{H}_\alpha = \{K(\cdot,\alpha)\}$ then minimality of $K(\cdot,\alpha)$ follows.

Setting $\varepsilon = 1/C(0)$, Theorem 27.12 yields $K(x_k,x_n) \ge \varepsilon/F(o,x_k)$ whenever $0 \le k \le n$. Therefore

$$F(x,x_k)K(x_k,\alpha) \ge \varepsilon\, K(x,x_k) \quad \text{for all } x \in X\,.$$

If $h \in \mathcal{H}^+$ is arbitrary then – using Lemma 27.5 – for all x

$$h(x) \geq F(x, x_k)\, h(x_k) \geq \varepsilon\, K(x, x_k)\, \frac{h(x_k)}{K(x_k, \alpha)}\,. \tag{27.16}$$

Now let $h \in \mathcal{H}_\alpha$, and apply (27.16) to $h' = K(\cdot, \alpha) - h$. Then

$$h'(x) \geq \varepsilon\, K(x, \alpha) \limsup_{k \to \infty} \frac{h'(x_k)}{K(x_k, \alpha)}\,.$$

As $\inf_X \big(h'/K(\cdot, \xi)\big) = 0$, we must have $\lim_k \big(h(x_k)/K(x_k, \alpha)\big) = 1$. We use this fact, and apply (27.16) to our $h \in \mathcal{H}_\alpha$. Letting $k \to \infty$, we infer $h \geq \varepsilon\, K(\,\cdot, \alpha)$. This holds for every $h \in \mathcal{H}_\alpha$.

Set $c_n = \varepsilon\big(1 + (1 - \varepsilon) + \cdots + (1 - \varepsilon)^n\big)$. We show inductively that $h \geq c_n\, K(\,\cdot, \alpha)$ for all $n \geq 0$. This is true for $n = 0$. Suppose it holds for $n-1$. Then the function $\frac{1}{1-c_{n-1}}\big(h - c_{n-1}\, K(\,\cdot, \alpha)\big)$ is also an element of $\mathcal{H}_\alpha$ and $\geq \varepsilon\, K(\,\cdot, \alpha)$. This yields $h \geq \big(c_{n-1} + \varepsilon(1 - c_{n-1})\big) K(\,\cdot, \alpha) = c_n\, K(\,\cdot, \alpha)$. Letting $n \to \infty$, we get $h \geq K(\,\cdot, \alpha)$. Therefore $h = K(\,\cdot, \alpha)$ for every $h \in \mathcal{H}_\alpha$. This concludes the proof of minimality of $K(\,\cdot, \alpha)$, and completes Step 2 and thus also Step 1.

At this point follows – without any change – the old Step 2, which now becomes Step 3, after which the proof is complete. (The old Step 3 has been modified and incorporated what is now Step 2 above.)

Missing reference. It is unforgivable that in the Preface there is no reference to the following book.

Guivarc'h, Y., Keane, M., and Roynette, B.: *Marches Aléatoires sur les Groupes de Lie*, Lect. Notes in Math. **624**, Springer, Berlin, 1977.

Indeed, while I did not use any specific material from that volume in the present monograph, it documents an important phase in the development of the theory of random walks on groups – not discrete ones, but Lie groups.

CONTENTS

CHAPTER I

THE TYPE PROBLEM

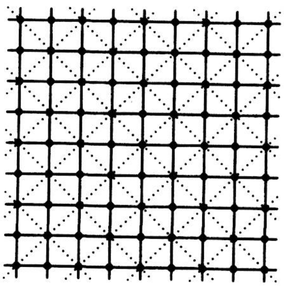
1. Basic facts

Before embarking on a review of the basic material concerning Markov chains, graphs, groups, etc., let us warm up by considering the classical example.

A. Pólya's walk

The d-dimensional grid, denoted briefly by $\mathbb{Z}^d$, is the graph whose vertices are integer points in d dimensions, and where two points are linked by an edge if they are at distance 1. A walker wanders randomly from point to point; at each "crossroad" (point) he chooses with equal probability the one among the $2d$ neighbouring points where his next step will take him, see Figure 1. Starting from the origin, what is the probability $p^{(2n)}(0,0)$ that the walker will be back at the $2n$th step? This is the number of closed paths of length $2n$ starting at the origin, divided by $(2d)^{2n}$. (The walker cannot be back after an odd number of steps.) For small dimensions, the solutions of this combinatorial exercise are as follows.

Figure 1: the grids $\mathbb{Z}$ and $\mathbb{Z}^2$



$d = 1$. Among the $2n$ steps, the walker has to make n to the left and n to the right. Hence

$$p^{(2n)}(0,0) = \frac{1}{2^{2n}}\binom{2n}{n} \sim C_1\, n^{-1/2}\,. \tag{1.1}$$

$d = 2$. Let two walkers perform the one-dimensional random walk simultaneously and independently. Their joint trajectory, viewed in $\mathbb{Z}^2$, visits only the set of points (i,j) with $i+j$ even. However, the graph with this set of vertices, and with two points neighbours if they differ by ± 1 in each component, is isomorphic with the grid $\mathbb{Z}^2$ and probabilities are preserved under this isomorphism. Hence

$$p^{(2n)}(0,0) = \left(\frac{1}{2^{2n}}\binom{2n}{n}\right)^2 \sim C_2\, n^{-1}. \tag{1.2}$$

$d = 3$. It is no longer possible to represent the random walk in terms of three independent random walks on $\mathbb{Z}$. In a path of length $2n$ starting and ending at the origin, n steps have to go north, east, or up. There are $\binom{2n}{n}$ possibilities to assign the n steps of these three types; the other n go south, west, or down. For each of these choices, i steps go north and i go south, j steps go east and j go west, $n-i-j$ steps go up and $n-i-j$ go down. Hence

$$p^{(2n)}(0,0) = \frac{1}{6^{2n}}\binom{2n}{n}\sum_{i+j\le n}\left(\frac{n!}{i!j!(n-i-j)!}\right)^2.$$

Consider the function $(x,y,z)\mapsto x!y!z!$ for $x,y,z\ge 0$. Under the condition $x+y+z=n$, it assumes its minimum for $x=y=z=n/3$, when n is sufficiently large. Hence

$$\begin{aligned} p^{(2n)}(0,0) &\le \frac{1}{6^{2n}}\binom{2n}{n}\frac{n!}{(n/3)!^3}\sum_{i+j\le n}\left(\frac{n!}{i!j!(n-i-j)!}\right) \\ &= \frac{1}{6^{2n}}\binom{2n}{n}\frac{n!}{(n/3)!^3}\,3^n \sim C_3\, n^{-3/2}. \end{aligned} \tag{1.3}$$

Indeed, for arbitrary dimension d, there are various ways to show that

$$p^{(2n)}(0,0) \sim C_d\, n^{-d/2}. \tag{1.4}$$

Now for the random walk starting at the origin, $\sum_n p^{(2n)}(0,0)$ is the expected number of visits of the walker back to the origin: this is infinite for $d = 1, 2$ and finite for $d \ge 3$. This drastic change of behaviour from two to three dimensions stands at the origin of our investigations.

B. Irreducible Markov chains

A *Markov chain* is (in principle) given by a finite or countable *state space* X and a stochastic *transition matrix* (or *transition operator*) $P = \big(p(x,y)\big)_{x,y\in X}$. In addition, one has to specify the starting point (or a starting distribution on X). The matrix element $p(x,y)$ is the probability of moving from x to y in one step. Thus, we have a sequence of X-valued random variables Z_n, $n \ge 0$, with Z_n representing the random position in X at time n. To model Z_n, the usual choice of probability space is the *trajectory space* $\Omega = X^{\mathbb{N}_0}$, equipped with the product σ-algebra arising from the discrete one on X. Then Z_n is the nth projection $\Omega \to X$. This describes the Markov chain starting at x, when Ω is equipped with the probability measure given via the Kolmogorov extension theorem by

$$\mathbb{P}_x[Z_0 = x_0, Z_1 = x_1, \ldots, Z_n = x_n] = \delta_x(x_0)p(x_0, x_1) \cdots p(x_{n-1}, x_n).$$

The associated expectation is denoted by $\mathbb{E}_x$. Alternatively, we shall call a Markov chain (random walk) the pair (X, P) or the sequence of random variables $(Z_n)_{n\geq 0}$. We write

$$p^{(n)}(x, y) = \mathbb{P}_x[Z_n = y].$$

This is the (x, y)-entry of the matrix power P^n, with $P^0 = I$, the identity matrix over X. Throughout this book, we shall always require that all states communicate:

(1.5) Basic assumption. *(X, P) is irreducible, that is, for every $x, y \in X$ there is some $n \in \mathbb{N}$ such that $p^{(n)}(x, y) > 0$.*

Next, we define the *Green function* as the power series

$$G(x, y|z) = \sum_{n=0}^{\infty} p^{(n)}(x, y)\, z^n, \quad x, y \in X, z \in \mathbb{C}. \tag{1.6}$$

(1.7) Lemma. *For real $z > 0$, the series $G(x, y|z)$ either diverge or converge simultaneously for all $x, y \in X$.*

Proof. Given $x_1, y_1, x_2, y_2 \in X$, by irreducibility there are $k, \ell \in \mathbb{N}$ such that $p^{(k)}(x_1, x_2) > 0$ and $p^{(\ell)}(y_2, y_1) > 0$. We have

$$p^{(k+n+\ell)}(x_1, y_1) \geq p^{(k)}(x_1, x_2)p^{(n)}(x_2, y_2)p^{(\ell)}(y_2, y_1)$$

and hence, for $z > 0$,

$$G(x_1, y_1|z) \geq p^{(k)}(x_1, x_2)p^{(\ell)}(y_2, y_1)z^{k+\ell}G(x_2, y_2|z). \qquad \square$$

As a consequence, all the $G(x, y|z)$ (where $x, y \in X$) have the same radius of convergence $\mathbf{r}(P) = 1/\rho(P)$, given by

$$\rho(P) = \limsup_{n\to\infty} p^{(n)}(x, y)^{1/n} \in (0, 1]. \tag{1.8}$$

This number is often called the *spectral radius* of P.

The *period* of P is the number $\mathrm{d} = \mathrm{d}(P) = \gcd\{n \geq 1 : p^{(n)}(x, x) > 0\}$. It is well known and easy to check that it is independent of x by irreducibility. If $\mathrm{d}(P) = 1$ then the chain is called *aperiodic*. Choose $o \in X$ and define $Y_j = \{x \in X : p^{(n\mathrm{d}+j)}(o, x) > 0 \text{ for some } n \geq 0\}$, $j = 0, \ldots, \mathrm{d} - 1$. This defines a partition of X, and x, y are in the same class if and only if $p^{(n\mathrm{d})}(x, y) > 0$ for some n. These are the periodicity classes of (X, P), visited by the chain $(Z_n)_{n\geq 0}$ in cyclical order. The restriction of P^{d} to each class is irreducible and aperiodic. See e.g. Chung [75] for these facts.