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APPLIED ANALYSIS

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To the Memory of the Six Million
Who Died for the Kiddush Hashem

PREFACE

FOR MANY YEARS the author has been engaged in studies of those fields of mathematical analysis that are of primary concern to the engineer and the physicist. That this area of "workable mathematics" did not receive the same attention during the 19th century as did the classical fields of analysis is perhaps the result of a historical misunderstanding. Until the time of Gauss and Legendre the "workable" methods of analysis received the closest attention of the best mathematicians. The brilliant discovery of the theory of limits changed the emphasis. Thenceforth it was considered satisfactory to design infinite approximation processes by which the validity of certain analytical results could be established, irrespective of whether the process used were feasible or not for a given problem.

It was then that the gradual separation of "pure" and "applied" mathematics occurred until we now have the "pure analyst," who pursues his ideas in a world of purely theoretical constructions, and the "numerical analyst," who translates the processes of analysis into machine operations.

In actual fact there is a large area between the two, which is not less analytical than the analysis of infinite processes but devoted to a different branch of analysis, namely, the analysis of *finite* algorithms. Here our objective is the analysis and design of finite processes which *approximate* the solution of an analytical problem. To design procedures which will effectively minimize the error in a small number of steps and which will estimate the error with sufficient accuracy is not a matter of *practical* interest only, but a matter of *scientific* interest as well. This book is largely devoted to such problems.

A few remarks concerning the manner of presentation may not be out of place. The author is not in favor of sacrificing rigor under the disguise that the applied scientist is interested only in the results and not in the more or less intricate procedures which lead to those results. The concepts and statements of mathematics are sharp and uncompromising and any "sloppy" presentation of a mathematical theorem disqualifies the formulation and throws doubt on the claimed result. It seems permissible, however, to state and prove a theorem under less exacting conditions than those in which the pure analyst is interested, if the gain achieved by this confinement is that the methods and results of mathematical investigations become presentable to the student of physics or scientific engineering in a language which is not overly strange to him. Furthermore, the author has the notion that mathematical formulas have their "secret life," behind their Golem-like appearance. To bring out the "secret life" of mathematical relations by an occasional narrative digression does not appear to him a profanation of the sacred rituals of formal analysis but merely an attempt to a more integrated way of understanding. The reader who has to struggle through a maze of "lemmas," "corollaries," and "theorems," can easily get lost in formalistic details, to the detriment of the essential elements of the results obtained. By keeping his mind on the principal points he gains in depth, although he may lose in details. The loss is not serious, however, since any reader equipped with the elementary tools of algebra and calculus can easily interpolate the missing details. It is a well-known experience that the only truly enjoyable and profitable way of studying mathematics is the method of "filling-in details" by one's own efforts. This additional work, the author hopes, will stir the reader's imagination and may easily lead to stimulating discussions and further explorations, on both the university and the research levels.

That a book of this nature cannot exhaust the subject without becoming unduly bulky, goes without saying. The broad subject of boundary-value problems, together with the theory of integral equations, had to be omitted, due to lack of space. But it is perhaps no exaggeration to say that the topics considered in each chapter are encountered almost daily by the engineer and physicist. A brief description of each chapter follows.

Chapter I. *Algebraic Equations.* The search for the roots of an algebraic equation is frequently encountered in vibration and flutter problems and in problems of static and dynamic stability. Some useful computing techniques, based on the "movable strip method," are discussed. The Bernoulli method with all its ramifications plays the central role, but the scanning of the unit circle for the separation of complex roots of nearly equal magnitude, and the method of reciprocal radii for stability questions are likewise of interest.

Chapter II. *Matrices and Eigenvalue Problems.* This chapter is devoted to a systematic development of the properties of matrices, with particular emphasis on those features which are most frequently encountered in industrial research.

Chapter III. *Large-Scale Linear Systems.* The advent of the electronic computer brings the iterative techniques for the solution of complicated boundary value problems and vibration problems into the foreground. This leads at once to the investigation of polynomial operations with matrices. While the general case of complex eigenvalues could not be included, the "spectroscopic method" of finding the real eigenvalues of large matrices and the corresponding method of solving large-scale linear equations is of such general usefulness—and so naturally tied up with the later chapter on harmonic analysis—that their treatment was hardly out of place. An additional treatment of a perturbation problem gives at least a partial answer to the complex eigenvalue problem by showing how an arbitrary complex eigenvalue and eigenvector can be obtained if we can start with a fairly good *first approximation* of the desired eigenvalue.

Chapter IV. *Harmonic Analysis.* The length of this chapter may be excused by the extraordinary importance of the Fourier series and its corollaries, the Fourier integral and the Laplace transform, in all problems of analysis. One might be tempted to paraphrase the famous saying of Victor Hugo that if he were asked to destroy all literature but keep one single book, he would preserve the Book of Job. Similarly, if we were asked to abandon all mathematical discoveries save one, we would hardly fail to vote for the Fourier series as the candidate for survival. This series has influenced the entire course of analysis, in both its theoretical and practical aspects, most profoundly. Moreover, its interconnection with other parts of analysis is so intimate that if we said "the Fourier series with all its implications," a considerable part of our classical analysis would be preserved.

For the purposes of engineering the orthogonality of the Fourier functions with respect to equidistant data is perhaps the most important single item. Accordingly the present chapter deals primarily with the interpolation aspects of the Fourier series, its flexibility in representing empirically given equidistant data with great ease. An important artifice is needed here to make the series applicable, viz., the subtraction of a linear trend which reduces the two boundary values to zero and permits the use of a pure sine series.

The remaining Gibbs oscillations, caused by the discontinuity of the *second* derivative, are too small to cause any harm.

An additional artifice, the application of the "σ-factors," counteracts the divergence-producing Gibbs oscillations if the series is *differentiated.* The same method smooths out the unpleasant Gibbs oscillations in the neighborhood of a discontinuity and in the representation of the delta-function.

In electric network analysis the methods of the Fourier transform and the Laplace transform have gained enormous impetus during the last few years. These transforms are not only theoretical devices for proving certain basic theorems, but are of fundamental importance for the practical construction of the input-output relation of linear networks. The interpolatory solution of the filter problem and a variety of methods for the inversion of the Laplace transform are discussed as problems of applied analysis which have very real technical significance. Finally, the frequently encountered "search for hidden periodicities" is dealt with and a numerical scheme developed which achieves greater independence of the various frequencies—and thus higher resolution power and higher accuracy—than the traditional schemes.

Chapter V. *Data Analysis.* The problem of the reduction of data and the problem of obtaining the first and even second derivatives of an empirically given function are constantly encountered in tracking problems, but also occur in similar form in ordinary curve-fitting problems. Two methods of smoothing are discussed: smoothing in the small and smoothing in the large. In the first case, local least-square parabolas are employed which lead to a certain weighted average of every observation with a few of its left and right neighbors. In the second case, a Fourier analysis is performed and the smoothing achieved by merely *truncating* the series at a judiciously chosen point. The latter technique provides us simultaneously with an analytical expression which represents all our data and interpolates them at any desired point of the interval. If it so happens that a close *polynomial* approximation is desired, a method is described which transforms the Fourier series into a polynomial series of strongest convergence. We thus avoid the pitfalls of equidistant Lagrangian interpolation and obtain a polynomial which fits our data with small and practically uniformly distributed errors.

Chapter VI. *Quadrature Methods.* Since the dawn of science the problem of integration has fascinated the scholar. Each scientific period added its own share of knowledge to the problem of quadrature. While Archimedes, who first introduced integration as an exact limit process, used only trapezoids with constantly decreasing sides for the purpose of quadrature, later ages refined the technique by operating with interpolating polynomials of higher order. The present chapter gives a survey of a variety of these methods. The Gaussian quadrature looms large as the most advanced of all quadrature methods. A slight modification of the method is described which makes it numerically more palatable by avoiding interpolation at irrational points. Moreover, the fundamental idea of the Gaussian method is translated to the case when only boundary values are at our disposal, viz., the value of the function and its derivatives up to a certain order at both endpoints of the interval. The resulting quadrature formula has strong convergence and can be used for the solution of boundary value problems and eigenvalue problems associated with ordinary differential equations. The method is demonstrated with the help of a few numerical examples.

Chapter VII. *Power Expansions.* The representation of functions by polynomials is an old art. However, to represent a function within a given interval by a few powers but with small error goes beyond the realm of the Taylor expansion and requires the theory of orthogonal function systems. There is in particular one special class of polynomials, the "Chebyshev polynomials," which assures stronger convergence than any other class of polynomials. In actual fact we go once more back to the Fourier series, since an expansion into Chebyshev polynomials is in reality nothing but a cosine series in a modified variable.

We can put these polynomials to good use in the problem of solving ordinary differential equations with coefficients which are rational functions of x. Since most of the fundamental transcendentals encountered in mathematical physics are definable in terms of such differential equations, we obtain rapidly convergent expansions for a large variety of functions by a simple technique. We terminate our series from the beginning to a finite polynomial of a given order. This necessitates an error term on the right side of the given differential equation. This error term is put proportional to a Chebyshev polynomial of properly chosen order. We now obtain simple recurrence relations from which the coefficients of the approximating polynomial can be determined. We may have to repeat this process several times and obtain the final result as a linear superposition of the composing polynomials. This "τ-method" is very helpful in putting the powers to work with maximum efficiency, whether our aim is to approximate an elementary function for operational purposes, or a transcendental function for evaluation purposes. The gain in reducing the error in comparison to that of the Taylor series (if the Taylor series exists at all) is always very considerable.

Acknowledgments. This book has grown through years of scientific thinking. In these years the author had the good fortune of innumerable discussions with colleagues and friends, which gave him the basic background on which to build. The number of people who helped the author in his endeavors is thus very great and their enumeration impossible. It will be more adequate to enumerate the institutions with which he was connected during the slow ripening of his thoughts.

1. During the author's memorable years at Purdue University (1931–45) Dr. W. Marshall of the Department of Mathematics and Dr. K. Lark-Horowitz of the Department of Physics organized a lecture course in "Approximation Methods of Analysis," where the author first came in touch with approximation problems. It was during those years that the author discovered for himself the outstanding properties of the Chebyshev polynomials, which had a decisive influence on his later scientific development.

2. During the national emergency the author spent the year 1943–44 at the Mathematical Tables Project, New York City, headed by Dr. A. N. Lowan, Director. The associations with an outstanding staff of numerical analysts were most gratifying.

3. During 1944–45 the author gave two lecture courses: "Engineering Applications of Rapidly Converging Series" and "Approximation Mathematics Course," under the auspices of the Physical Research Unit of the Boeing Airplane Company in Seattle, Washington. The excellently prepared mimeographed notes of these courses—compiled with the able assistance of Marius Cohn—form the basic core of the present book.

In 1946 Dr. C. K. Stedman, Head of the Physical Research Unit of the Boeing Airplane Company, invited the author to join his unit as a mathematical consultant and research engineer. The benefits he

derived from the daily discussions with an unusually select group of excellent physicists and electrical engineers cannot be measured.

4. At the invitation of Dr. J. H. Curtiss, then Chief, National Bureau of Standards, Washington, D. C. the author joined in 1949 the newly founded Institute for Numerical Analysis at the University of California, Los Angeles. Under the leadership of Dr. Curtiss, the Institute provided a scholarly atmosphere and colleagual associations on a level which had few parallels anywhere in the world. The generosity with which the Institute supported the author's scientific projects will remain in his memory with undiminished force.

5. At the invitation of the Dublin Institute for Advanced Studies, the author spent a memorable year (1952–53) in Dublin, Eire, in daily contact with the directors of the Institute, Professor E. Schroedinger and Professor J. L. Synge. The quadrature formula of § 22, Chapter VI, and its application to the solution of eigenvalue problems were developed during this year.

6. During the winter 1953–54 the author was affiliated with North American Aviation, Los Angeles, as a staff member of the Tabulating Department of Charles F. Davis, numerical analyst. A lecture course attended by a selected group of engineers led to memorable friendships and is reflected in Chapters II and III of this book. The "spectroscopic eigenvalue analysis" was developed during these months.

In enumerating the extraordinary opportunities with which his good fortune endowed him in his scientific life, the author should not fail to mention the splendid assistance he received in the numerical documentation of his mathematical endeavors. For the computation of the numerical examples of the book he is primarily indebted to Miss Mary Ellen Russell, Research Assistant of the Physical Research Unit, Boeing Airplane Company, Seattle, Washington, while the Appendix Tables were principally prepared by Miss Lillian Forthal, INA, National Bureau of Standards, Los Angeles, California.

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BIBLIOGRAPHY

The following books are recommended for collateral or more advanced reading. They represent a selective bibliography of books, written in English, whose topic is similar or related to that of the present book. If important sources have been overlooked, this did not occur intentionally. The Courant-Hilbert and Whittaker-Watson books are irreplaceable sources of information. In the field of numerical analysis the books of Milne, Scarborough, and Hartree are basic.

{1} COURANT, R. and D. HILBERT, *Methods of Mathematical Physics*, Vol. I (Interscience Publishers, New York, 1953).

{2} DOHERTY, R. E. and E. G. KELLER, *Mathematics of Modern Engineering*, Vols. I and II (John Wiley & Sons, Inc., New York, 1936 and 1942).

{3} DWYER, P. S., *Linear Computations* (John Wiley & Sons, Inc., New York, 1951).

{4} HARTREE, D. R., *Numerical Analysis* (Clarendon Press, Oxford, 1952).

{5} HOUSEHOLDER, A. S., *Principles of Numerical Analysis* (McGraw-Hill Book Company, Inc., New York, 1953).

{6} JAEGER, J. C., *Applied Mathematics* (Clarendon Press, Oxford, 1951).

{7} KARMAN, TH. v. and M. A. BIOT, *Mathematical Methods of Engineering* (McGraw-Hill Book Company, Inc., New York, 1940).

{8} MILNE, W. E., *Numerical Calculus* (Princeton University Press, Princeton, N. J., 1949).

{9} MURNAGHAN, T. D., *Introduction to Applied Mathematics* (John Wiley & Sons, Inc., New York, 1948).

{10} PIPES, L. A., *Applied Mathematics for Engineers and Physicists* (McGraw-Hill Book Company, Inc., New York, 1946).

{11} SCARBOROUGH, J. B., *Numerical Mathematical Analysis* (Johns Hopkins Press, Baltimore, 1950).

{12} SCHELKUNOFF, S. A., *Applied Mathematics for Engineers and Scientists* (D. Van Nostrand Company, Inc., New York, 1948).

{13} SMITH, L. P., *Mathematical Methods for Scientists and Engineers* (Prentice-Hall, Inc., Englewood Cliffs, N. J., 1953).

{14} WHITTAKER, E. T. and G. ROBINSON, *The Calculus of Observations* (Blackie & Son, Glasgow, 1940).

{15} WHITTAKER, E. T. and G. N. WATSON, *A Course of Modern Analysis* (Cambridge University Press, London, 1935).

Books of a more specific character are listed at the end of each chapter. References in braces { } refer to the books of the general Bibliography, those in brackets [ ] to the books and articles of the chapter bibliographies.

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