CONTEMPORARY MATHEMATICS

490

Symmetry in Mathematics and Physics

Conference in Honor of V. S. Varadarajan's 70th Birthday
January 18–20, 2008
University of California, Los Angeles, California

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Symmetry in Mathematics and Physics

Preface

The articles in this collection mainly grew out of the talks given at a Conference held at UCLA in January 2008, which honored V. S. Varadarajan on his 70th birthday. The main theme of the Conference was symmetry in mathematics and physics. More precisely, the talks at the conference were dedicated to the interplay between geometry, group theory, and fundamental physics. In addition to the speakers there were a number of doctoral and post doctoral fellows including several students of Varadarajan who had worked under him on these topics throughout his career.

Varadarajan's work over the past 50 years represents a broad spectrum of mathematics but its main emphasis has been on symmetry in mathematics and mathematical physics, broadly interpreted. Some of his significant achievements are: development of the infinitesimal method in the theory of infinite dimensional representations of real semi simple Lie algebras; Fourier transform theory in the complex domain on Riemannian symmetric spaces; theory of local moduli for ordinary meromorphic linear differential equations at an irregular singularity; theory of unitary representations of super Lie groups and the classification of super particles; and more recently, studies on the physics associated to non-archimedean space-time.

The relevance of the representation theory of Lie groups and Lie algebras to the physics of elementary particles and fields has been known for a very long time, going back to the famous 1939 paper of E. P. Wigner on the representations of the Poincare group. Since then this link between representation theory and physics has deepened enormously, and includes quantum field theory and conformal field theory. Then something marvelous happened. In the 1970's the physicists created a new extension of geometry where the underlying manifolds acquired anti-commuting coordinates in addition to the usual commuting ones, reflecting the Fermionic structure of matter. This introduced supergeometry and super Lie groups into the mix and made the connection between geometry and physics much richer. Together with his students, he has made many important contributions to this area.

It thus seemed appropriate to have a conference at UCLA devoted to some of these themes. The Conference turned out to be very exciting and stimulating because of the contributions of the participants who came from the United States and abroad. Most of the articles in this volume are thus naturally concerned with the above-mentioned themes: representations of finite and infinite dimensional Lie groups and Lie algebras, super Lie groups and supergeometry, which are at the interface of mathematics and fundamental particle physics, and supersymmetry. The discussions on supergeometry and supersymmetry are especially relevant at this time since some of the experiments at the Large Hadron Collider at CERN may help determine whether supersymmetry is a feature of the world of elementary particles. A few of the articles are in probability and the foundations of quantum mechanics, areas in which Varadarajan worked early in his career.

The scientific organizing committee consisted of S. R. S. Varadhan (Chair), E. Beltrametti, T. J. Enright, S. Ferrara, K. R. Parthasarathy, and N. R. Wallach. The conference was funded by a generous grant from NSF, from a grant from the Goldman-Sachs corporation, and a matching contribution from a private donor. We are very grateful to these sources for their generosity, which made the conference go on in a very smooth manner.

The editors wish to express their thanks to many people whose efforts made this conference a success, including all the participants. They wish to thank IPAM for permission to use their facilities for the conference; Professors Christoph Thiele and Robert Steinberg for inaugurating the conference; the department of mathematics at UCLA for providing help at all stages of the conference and for organizing the web site, especially Babette Dalton, Robert Amodeo, and Natasja Saint-Satyr; and Christine Thivierge of the AMS for her editorial assistance in producing this collection. The editors are also grateful to T. Kibble and Imperial College Press for permission to reprint B. Zumino's paper "Supersymmetry: A Personal View", which appears by their courtesy in these pages.

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The Role of Weak Convergence in Probability Theory

S.R.S.Varadhan

1. Introduction

The concept of limit theorems in probability goes way back. The first limit theorem, the weak law of large numbers was proved by Jacob Bernoulli [**1**] in the early eighteenth century. This was quickly followed by De Moivre [**3**] who proved the central limit theorem establishing the approximation of the binomial distribution by a normal distribution. Further work by, Laplace, Gauss, Levy, Khintichine, Kolmogorov, Gnedenko and others put limit theorems at the center of probability theory [**11**].

The connection between random walks and Brownian motion was understood by Levy and others along with the idea that distributions of quantities like the maximum etc, based on random walks, converges to the corresponding distributions derived from Brownian motion. Doob [**6**] formulated this more precisely in his paper on 'Heuristic approach to the Kolmogorov-Smirnov Theorems'. Donsker [**5**], in his thesis, established the first general theorem to the effect that Doob's heuristic proof can in fact be justified. However his approach was too dependent on finite dimensional approximations.

At this point the study of stochastic processes as probability distributions on function spaces began. Contributions were made by LeCam [**13**] in the United States, Kolmogorov [**12**], Prohorov [**15**], Skorohod [**17**] and others in USSR, as well as Varadarajan [**20**] in India. Alexandrov in the 1940's had studied set functions on topological spaces and now powerful techniques from functional analysis could be used to study stochastic processes as measures on function spaces. A random walk or any stochastic process induces a probability distribution on the space of paths. By interpolation or some such simple device both the approximating and the limiting distributions can be put on the same space of paths. The question then reduces to the investigation of the convergence of a sequence μ_n of probability measures on a space X of paths to a limit μ. It is clear that the measures μ_n, in the case of random walks, look qualitatively different from Brownian paths and hence $\mu_n \perp \mu$. It is not going to be true that $\mu_n(A) \to \mu(A)$ for all measurable sets $A \in X$.

Functional analysis now provides a useful window. The space X of paths comes with a topology. A probability measure μ defines a normalized non-negative linear functional

$$\Lambda_\mu(f) = \int f(x)\mu(dx)$$

on $\mathbf{B} = C(X)$ and they form a convex subset $\mathcal{M}$ in the dual $\mathbf{B}^*$ of $\mathbf{B}$. The natural weak$*$ topolgy on this subset is called 'weak convergence' by probabilists, and has proved to be a very useful tool. The following questions arise naturally and were investigated.

1. Which linear functionals $\Lambda(f)$ have the above representation as Λ_μ for some probability measure μ, i.e. when does $\lambda \in \mathcal{M}$ and when is μ uniquely determined by Λ?

2. What can one say about the space $\mathcal{M}$ as a topological space?

3. Is it metrizable? What are its compact subsets?

Viewed in this manner the classical limit theorems tell us that if we place random walks as well as Brownian Motion on the same space $X = C[0, T]$ of continuous paths, then the only possible limit for the measures μ_n coming from random walks is the Wiener measure μ. Weak convergence as elements in $C(X)^*$ would provide justification for the convergence of distributions of continuous functionals on X under μ_n to the corresponding distribution under μ. The issue then is that of compactness.

Prohorov, Skorohod and others in the Russian school worked mostly under the assumption that X is a complete separable metric space. They obtained characterizations of compact subsets of $\mathcal{M}$, and provided useful general sufficient conditions to verify compactness in several useful function spaces. Varadarajan had independently worked out similar results, in a more general context, in his thesis during 1956-57 at the Indisn Statistical Institute. These tools were used by Parthasarathy, Ranga Rao and Varadhan [**14**] , [**21**] to study limit theorems in different contexts.

2. The Martingale Problem.

The work of Stroock and Varadhan [**18**], [**19**] on the Martingale approach to the study of Markov process is also motivated by these considerations. In approximating diffusion processes by Markov chains we start with an approximation that is valid at the infinitesimal level. If $h > 0$ is the discretized time unit and $\pi_h(x, dy)$ is the single step transition probability, it is natural to assume that

$$\int [f(y) - f(x)]\pi_h(x, dy) = h(Lf)(x) + o(h)$$

where L is the generator of the semigroup associated with the limiting process. Our aim is to show $v_n(x) = \int f(y)\pi_h^n(x, dy) \simeq (e^{n\,h\,L} f)(x)$. The standard analytical method is to to solve the evolution equation

$$u_t = Lu, u(0, x) = f(x)$$

and estimate the difference $\Delta_n(x) = v_n(x) - u(n\,h, x)$. Let $nh = t$.

$$\begin{aligned}\int u(k\,h, y)\pi_h(x, dy) &\simeq u(k\,h, x) + h(Lu(k\,h, \cdot))(x) + o(h)\\ &= u(k\,h, x) + hu_t(k\,h, x)(x) + o(h)\\ &= u((k+1)\,h, x) + o(h)\end{aligned}$$

Hence

$$\left|u(n\,h,x) - \int u(0,y)\pi_h^n(x,dy)\right| \le n\,o(h) = o(1)$$

and the proof depends on the regularity of the solution $u(t,x)$. The actual random paths of the Markov chain did not play any role. Only the iterates of π_h were needed.

The martingale method, on the other hand, starts with the measures $P_{h,x}$ on a function space of paths, that corresponds to the given Markov chain starting from x and shows that sequence has a limit P_x. Then if $nh = t$,

$$v_n(x) = E^{P_{h,x}}[f(x(t)] \to E^{P_x}[f(x(t)] = u(t,x)$$

To achieve this one needs to verify compactness and then characterize the possible limit uniquely. It is done as the unique measure on $C[0,T]$ (or $D[0,T]$ which is a space of paths that admits simple jumps), with certain properties:

$$P[x(0) = x] = 1$$

and

$$f(x(t)) - f(x(0) - \int_0^t (Lf)(x(s))ds = Z_f(t)$$

is a martingale with respect to $(\Omega, \mathcal{F}_t, P)$ for a wide class of functions f. From the definition of a Markov chain, if P_h is the probability distribution of the chain and

$$P_h[x(0) = x] = 1$$

then

$$f(x(n\,h)) - f(x(0)) - \sum_{j=0}^{n-1} \int [f(y) - f(x(j\,h))]\pi_h(x(j\,h), dy) = Z_f^h$$

is a martingale. One now uses ideas from weak convergence to show that $\{P_h\}$ is compact and that any limit satisfies the properties.

3. Large Deviations

In limit theorems one is primarily interested in the behavior of $P_n(A)$ as $n \to \infty$. We look at the larger problem of weak convergence of P_n and establish such convergence to the limit P, in a some topological space X that contains A as a subset. If A is a continuity set for P, i.e $P[\delta A] = 0$, then $P_n(A) \to P(A)$.

In large deviations, we are dealing with a situation where P_n tends to a distribution that is degenerate at some point, i.e. $P_n \to \delta_{x_0}$ for some $x_0 \in X$. Then if $x \notin \bar{A}$, $P_n(A) \to 0$ and we wish to know how fast. In particular we expect the rate to be exponential and we wish to examine

$$\lim_{n\to\infty} \frac{1}{n} \log P_n(A) = -c(A)$$

It will turn out that

$$c(A) = \inf_{x \in A} I(x)$$

for some non-negative function $I(x)$ with $I(x_0) = 0$.

In the theory of weak convergence it is established that in metric spaces weak convergence is equivalent to

$$\limsup_{n\to\infty} P_n(C) \le P(C) \qquad \text{for closed sets} \quad C$$
$$\liminf_{n\to\infty} P_n(G) \ge P(G) \qquad \text{for open sets} \quad G$$

and for continuity sets i.e sets with $P(A^o) = P(\bar{A})$, one has

$$\lim_{n\to\infty} P_n(A) = P(A)$$

An important criterion for compactness of $\{P_n\}$, is the uniform tightness condition: given $\epsilon > 0$, there is a compact set K_ϵ such that

$$\inf_n P_n(K_\epsilon) \ge 1 - \epsilon$$

In large deviation theory a sequence P_n is said to satisfy the large deviation principle with the rate function $I(x)$ if

$$\limsup_{n\to\infty} \frac{1}{n} \log P_n(C) \le -\inf_{x\in C} I(x) \qquad \text{for closed sets} \quad C \tag{3.1}$$

$$\liminf_{n\to\infty} \frac{1}{n} \log P_n(G) \ge -\sup_{x\in G} I(x) \qquad \text{for open sets} \quad G \tag{3.2}$$

$I(x) : X \to [0,\infty]$ is a lower semi continuous function. There is a similar tightness condition.

1. $I(x)$ has compact level sets, i.e $C_\ell = \{x : I(x) \le \ell\}$ is compact for each $\ell < \infty$.

2. For any ℓ there is a compact set K_ℓ such that

$$\sup_n P_n(K_\ell^c) \le e^{-n\ell}$$

Just as weak convergence implies

$$\lim_{n\to\infty} P_n(A) = P(A)$$

for continuity sets, in large deviation theory it follows easily from (3.1) and (3.2) that for sets A such that

$$\inf_{x\in A^o} I(x) = \inf_{x\in A} I(x) = \inf_{x\in\bar{A}} I(x)$$

we will have

$$\lim_{n\to\infty} \frac{1}{n} \log P_n(A) = -c(A) = -\inf_{x\in A} I(x)$$

In weak convergence we will have

$$\lim_{n\to\infty} \int f(x)dP_n = \int f(x)dP$$

for bounded continuous functons f. In large deviation theory for such functions the analogous result is

$$\lim_{n\to\infty} \frac{1}{n} \log \int e^{n\,f(x)} dP_n = \sup_x [f(x) - I(x)]$$

In weak convergence when $f_n \to f$, in order to make sure that $\int f_n(x)dP_n \to \int f(x)dP$ when P_n converges weakly to P, one assumes the uniform convergence of f_k to f on compact sets. This is used together with tightness to control

$$\limsup_{k\to\infty} \limsup_{n\to\infty} P_n[|f_k - f| \geq \epsilon] = 0$$

for every $\epsilon > 0$. In large deviation theory super exponential estimates play a similar role. For every $\epsilon > 0$,

$$\limsup_{k\to\infty} \limsup_{n\to\infty} \frac{1}{n} \log P_n[|f_k - f| \geq \epsilon] = -\infty \tag{3.3}$$

The methods used in large deviation theory are often similar to the ones used in studying weak convergence of probability measures. There are several sources [**22**], [**4**], [**2**] that describe these. Let us look at an example.

Schilder's theorem [**16**]. P_n is the distribution of $x(t) = \frac{1}{\sqrt{n}}\beta(t)$, where β is the standard Brownian motion. There is a large deviation principle on $C[0,1]$ with rate function

$$I(f) = \frac{1}{2}\int_0^1 [f'(t)]^2 dt$$

on functions f with $f(0) = 0$ that have a square integrable derivative. Otherwise $I(f) = \infty$.

The proof proceeds by approximation. If $x_k(\cdot)$ is the piecewise linear approximation with intervals $\frac{1}{k}$.

$$P_n[\|x_k(\cdot) - x(\cdot)\| \geq \epsilon] \leq C\, n\, exp[-\frac{n\, k\, \epsilon^2}{2}].$$

This provides the estimate (3.3) and enables us to interchange the limits on k and n. In a recent book [**7**] Ellis and Dupuis emphasize the weak convergence aspect of large deviations. If one wants to prove a large deviation upper bound for P_n with rate function $I(x)$, it suffices to show that when ever $Q_n << P_n$ and $Q_n \to \delta_x$ weakly, then

$$\liminf_{n\to\infty} \frac{1}{n} H(Q_n|P_n) \geq I(x).$$

Here $H(Q|P)$ is the relative entropy $\int \frac{dQ}{dP} \log \frac{dQ}{dP}\, dP$.

4. Scaling Limits

In studying scaling limits of large systems of interacting processes ideas from weak convergence play a crucial role. For simplicity consider as in [**8**], a family $\{x_i(t)\}$ of processes indexed by points $i = 1, 2, \ldots, N$ arranged (periodically) on the unit circle that satisfy the stochastic differential equation

$$dx_i(t) = [\phi(x_{i-1}(t)) - 2\phi(x_i(t)) + \phi(x_{i+1}(t))]dt + [d\beta_{i,i+1}(t) - d\beta_{i-1,i}(t)].$$

Assume initially that

$$\frac{1}{N}\sum_i \delta_{\frac{i}{N}} x_i(0) = \mu_N(0) \to u_0(x)dx$$

in probability. Then show that

$$\frac{1}{N}\sum_i \delta_{\frac{i}{N}} x_i(N^2 t) = \mu_N(N^2 t) \to u(t,x)dx$$

where $u(t,x)$ satisfies a certain partial differential equation with $u(0,x) = u_0(x)$. Let P_N be the distribution of the process $\mu_N(N^2\cdot)$ on the space $C[[0,T];\mathcal{M}(S^1)]$. Then,

1. P_N is tight.

2. If P is any limit then P is supported on paths $\mu(t)$, that satisfy

(i) they have a density for every t, i.e $\mu(t,dx) = u(t,x)dx$

(ii) if $\psi' = \phi$ then with

$$h(x) = \sup[\theta x - \log \int e^{\theta y - \psi(y)} dy]$$

$u(t,x)$ is a weak solution of

$$u_t = [h(u(t,x))]_{xx}$$

that satisfies

$$\int_{S^1} \int_0^T [[h'(u(t,x))]_x]^2 dxdt < \infty$$

(iii) $u(0,x) = u_0(x)$

3. The solution satisfying (i), (ii) and (iii) is unique.

The measure P therefore has to be the δ-measure at this unique solution.

5. Large Deviations for the Simple Exclusion Process.

The totally asymmetric simple exclusion model (TASEP) is a Markov process of interacting particles [**10**]. These particles live on Z and are restricted to at most one particle per site. The current state then is a map $\eta : Z \to \{0,1\}$. If there is a particle at site x then $\eta(x) = 1$, otherwise it is $\eta(x) = 0$. The generator of the process is given by

$$(\mathcal{L}f)(\eta) = \sum_z \eta(z)(1-\eta(z+1))[f(\eta^{z,z+1}) - f(\eta)]$$

which means that particles wait for a random exponential time at the end of which they try to jump to the next site on its right. They jump if the site is free. If not they wait again for the next opportunity. If the density of particles is not too high, then they march to the right, in a haphazard fashion. We rescale space by $\frac{1}{N}$ and speed up time by a factor of N. The density profile $\rho(t,x)$ that provides a macroscopic view, at time Nt is defined by

$$\frac{1}{N}\sum_z J(\frac{z}{N})\eta(Nt,z) \to \int J(x)\rho(t,x)dx.$$

Given $\rho(0,x) = \rho_0(x)$ at time $t=0$, for $t>0$, $\rho(t,x)$ is determined as the weak solution of

$$\rho_t + [\rho(1-\rho)]_x = 0$$

with initial condition $\rho(0,x) = \rho_0(x)$. However the weak solution is not unique and an 'entropy condition' has to be imposed in order to make the solution unique. If ϕ is a convex function and h is determined by $h'(r) = \phi'(r)(1-2r)$, then for smooth solutions we will have

$$[\phi(\rho)]_t + [h(\rho)]_x = 0.$$