FELIX KAUFMANN

# THE INFINITE IN MATHEMATICS

*Logico-mathematical writings*

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# THE INFINITE IN MATHEMATICS

# VIENNA CIRCLE COLLECTION



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FELIX KAUFMANN (1895–1949)

# INTRODUCTION

The main item in the present volume was published in 1930 under the title *Das Unendliche in der Mathematik und seine Ausschaltung*. It was at that time the fullest systematic account from the standpoint of Husserl's phenomenology of what is known as 'finitism' (also as 'intuitionism' and 'constructivism') in mathematics. Since then, important changes have been required in philosophies of mathematics, in part because of Kurt Gödel's epoch-making paper of 1931 which established the essential incompleteness of arithmetic. In the light of that finding, a number of the claims made in the book (and in the accompanying articles) are demonstrably mistaken. Nevertheless, as a whole it retains much of its original interest and value. It presents the issues in the foundations of mathematics that were under debate when it was written (and in some cases still are); and it offers one alternative to the currently dominant set-theoretical definitions of the cardinal numbers and other arithmetical concepts.

While still a student at the University of Vienna, Felix Kaufmann was greatly impressed by the early philosophical writings (especially by the *Logische Untersuchungen*) of Edmund Husserl. He was never an uncritical disciple of Husserl, and he integrated into his mature philosophy ideas from a wide assortment of intellectual sources. But he thought of himself as a phenomenologist, and made frequent use in all his major publications of many of Husserl's logical and epistemological theses. He had been a student of the legal philosopher Hans Kelsen at the University, received the doctorate in law in 1920 and the doctorate in philosophy two years later, and on Kelsen's recommendation was appointed in the latter year to the unsalaried post of *Privatdozent* of the philosophy of law in the juridical faculty of that institution.

To earn a living, Kaufmann sought employment in business, and eventually became the manager of the Austrian branch of the Anglo-Iranian Oil Company. Nevertheless, by the time he came to write the present book he had published three others on the philosophy of law. He undertook in them to recast Kelsen's 'pure theory of law' by substituting for its neo-Kantian assumptions, which Kaufmann found unsatisfactory, a phenomenological epistemology. He found time to be a fairly regular attendant at the meetings of the *Wiener Kreis* (or the *Vienna Circle*), a discussion group of philosophers and scientists organized by Moritz Schlick after he joined the faculty of the University as professor of philosophy, which eventually acquired international repute as the progenitor of logical positivism (or logical empiricism, as the movement was also called). However, although Kaufmann was in wholehearted sympathy with many of its attitudes – especially with its stress on clarity and logical rigor in the conduct of philosophical inquiry – he objected to being counted as a logical positivist, and he saw himself as constituting the loyal opposition to the atomistic empiricism of the *Kreis*. Kaufmann's indebtedness to Husserl continued to be exhibited in his *Methodenlehre der Sozialwissenschaften* published in 1936, as well as in its completely rewritten version published in 1944 with the title *Methodology of the Social Sciences*. The latter work, written in English after he left Hitler's Vienna to join the Graduate Faculty of the New School for Social Research in New York City, also reveals the influence on his thought of John Dewey's logical theory when he became familiar with it in the United States. Kaufmann died unexpectedly in 1949 at the age of 54 years.

This summary account of Kaufmann's career makes evident the unusually broad scope of his active scholarly interests. This breadth flowed directly from his conception of the general task of philosophy. As he saw it, that task is to provide an ongoing critique of knowledge by articulating the logical conditions discourse must satisfy to be meaningful, making explicit the rules governing the acceptance and rejection of beliefs, and thereby producing the intellectual tools for clarifying and evaluating unsettled issues in various branches of inquiry. In consequence, the central objective of his wide-ranging studies was to make manifest the principles men employ when they succeed in making their experience intelligible, and to assess in the light of those principles disputed claims to knowledge in a number of special disciplines.

Kaufmann's pursuit of these objectives was controlled by a variety of special assumptions, commonly though not exclusively made by phenomenologists. The most inclusive of these, the Principle of Phenomenological Accessibility, asserts that whatever has a locus in any realm of being is 'accessible to cognition', so that there is nothing inherently unknowable or incapable of precise analysis. Although it is not entirely clear what it means to be 'accessible to cognition', Kaufmann used this Principle to show that concepts apparently referring to things transcending all possible experience either have no function in the acquisition and edifice of knowledge (and are therefore eliminable), or have meanings specifiable in terms of experimentially identifiable procedures. Kaufmann's approach to the task of clarifying ideas has much in common with the 'operationalism' of P. W. Bridgman and other instrumentalist thinkers, though he rejected the sensationalistic epistemology to which Bridgman subscribed.

It is this 'procedural approach' to the analysis of concepts that characterizes Kaufmann's philosophy of mathematics and the discussions in the present work. The reform movement in the 19th century known as 'the arithmetization of mathematics' sought to remove serious obscurities and confusions in various branches of mathematical analysis by redefining all the concepts used in them (such as the notions of imaginary and irrational number, continuity, or the derivative of a function) in terms of the familiar arithmetical operations upon integers; and in consequence, the assumption that dubious 'entities' such as infinitesimals are needed for differentiating or integrating functions was shown to be unnecessary. However, with the development of set-theory during the second half of the century – the theory came to be regarded eventually as the foundation for the rest of mathematics – 'objects' of a new sort were introduced into the subject (such as nondenumerably infinite classes and transfinite numbers), which many outstanding mathematicians, among others E. Borel, L. E. J. Brouwer, and H. Weyl, believed were as questionable as was the assumption of infinitesimals. (But the notion of infinitesimals has been placed on secure foundations during the past twenty five years, so that the 19th-century objections to infinitesimals cannot be validly raised against the revised notion. Infinitesimals have become respectable.) Kaufmann shared this belief, and the present book is his attempt to show that contrary to appearances mathematics nowhere requires the notion of an 'actual infinite', and that the standard arguments for the 'existence' of various orders of infinity are fallacious.

Kaufmann subscribed to the familiar Leibnizian distinction between necessary truths of reason and contingent truths of empirical fact. The former are certifiable by examining the meanings (or connotations) of the terms contained in them; the latter are based on the outcome of observation or experiment, and are in principle always corrigible. In consonance with this fundamental dichotomy, he followed Husserl in distinguishing between two sorts of universal statements, called 'specific' and 'individual'. A *specific universal* (such as the statement 'All prime numbers greater than two are odd') is said to assert that a specified relation holds between *concepts*, so that deductive logic allegedly suffices to establish the truth or falsity of the statement. On the other hand, an *individual universal* (such as the statement 'All the animals exhibited in the Milwaukee Zoo during the current year weigh more than two pounds') is characterized as being, in effect, the conjunction of a finite number of singular statements about certain individuals in specified spatio-temporal region, so that the truth-value of the universal can be determined by examining those individuals seriatim. It is therefore alleged that it is a blunder to suppose that the sense of the particle 'all' in specific universals is the same as the sense of 'all' in individual universals, or that specific and individual universals can be construed in identical ways. In Kaufmann's judgment, however, it is just such a conflation of two radically different meanings of 'universal statement' which is at the bottom of the allegedly mistaken beliefs that cardinal numbers are logically prior to the ordinals, and that there are infinite classes whose members are non-denumerable.

Kaufmann's book must be consulted for the details of his argument for these conclusions, and only so much of its salient features will be mentioned as is needed to make intelligible some comments on several of his major claims. In contrast to the Frege-Russell definition of the cardinal numbers as classes (or sets) of similar classes, Kaufmann defined them as invariants of counting processes – that is, in whatever order the members of a collection are matched with the members of some serially ordered set of standard items, such as the numerals, the last member of the collection to be matched will always correspond to the same numeral. Accordingly, though the cardinal number of a collection is independent of *any one* order in which its members are counted, it is not independent of *all* such orders, so that ordinal numbers are logically prior to the cardinals. On the other hand, the ordinal numbers themselves are defined, in a manner substantially in agreement with the Peano axioms for arithmetic, as the formal structures embedded in counting processes that have no fixed termination. In consequence, the phrase 'the infinite series of the integers' must not be taken to denote an 'actual infinite totality', as if statements about all the integers were individual universals. On the contrary, the phrase is said to be a term that enters into specific universals, and to signify the formal serial structures that are distinctive of processes of counting. In the case of a finite collection of items, it makes good sense to talk about the class of all its sub-classes, for not only can each of these sub-classes be 'constructively' defined (that is, a rule or 'law' can be stated for obtaining each of them), but the set (or class) of *all* these sub-classes can also be constructively defined. However, on this construal of the term 'set' or 'class' (according to which, quoting Kaufmann, 'an infinite set is nothing but a law'), the expression 'the class of all sub-classes of the class of all integers' is meaningless. For the so-called 'class' mentioned in the expression is not, and cannot be, constructively defined, so that it is also meaningless to assert that 'the members of this class' are non-denumerable. Moreover, Kaufmann maintained that his definition of the integers established not only the *consistency* of Peano's axioms (that is, that there is no arithmetical statement such that both the statement and its denial are derivable from the axioms), but also their *completeness* (that is, that there is no arithmetical statement such that neither the statement nor its negation is derivable from the axioms).

Kaufmann's rejection of Cantor's 'diagonal proof' that the 'totality' of the real numbers (or the continuum) is non-denumerable is based on the same considerations that led him to reject as meaningless the expression 'the class of all sub-classes of the set of integers'. For the diagonal proof proceeds on the assumption that it makes sense to talk about the totality of all the reals, an assumption Kaufmann denied on the ground that the alleged totality is not constructively definable, and that the assumption confounds the sense of 'all' in specific universals with the sense of the particle in individual ones. According to him, what the diagonal proof does establish is that for any given denumerable sequence (i.e., for any sequence specified by some determinate rule for constructing its members) of denumerable sequences of integers (which are also specified by some constructive rule), another sequence of integers can be defined (i.e., another rule of construction can be formulated) which is not included in the initial rules of construction. But it does not follow from this conclusion, so he maintained, that the real numbers (or rules for constructing sequences of integers) form a totality and that they are non-denumerable.

Although Kaufmann was not alone in defining cardinal numbers in terms of the ordinals or in rejecting as absurd the notion of non-denumerable classes, the reasons he gave for these views were in considerable measure his own. Moreover, unlike many who arrived at similar conclusions (notably the mathematician Brouwer), he believed that the formal structures investigated by mathematicians are discoveries rather than human creations, and that the constructive intuitionism to which he subscribed does not require the rejection as false of any principles of classical logic (such as the principle of excluded middle).

Kaufmann was unquestionably correct in holding that the cardinal numbers *can* be defined in terms of the ordinals. The mode of defining the cardinals he proposed has some clear advantages over the alternative Frege-Russell procedure – for example, his method makes more evident than does the set-theoretical definition the function of cardinal numbers in normal every-day and scientific practice. However, he did not recommend his way of defining the cardinals for such pragmatic reasons. He did so because he believed that the set-theoretical definition is fundamentally unsound and involves a serious blunder. But it is not obvious that Kaufmann succeeded in showing this to be the case, and it is therefore appropriate to ask whether he did in fact accomplish this.

The premises on which he based the conclusion that the set-theoretical definition is unsound include the assumption – let us grant it without discussion for the sake of the argument – that the definition involves the interpretation of specific universals as if they were individual universals. Kaufmann's case against the set-theoretical definition then depends on whether it is a hopeless error, as he believed, to suppose that terms occurring in specific universals have *extensions* and that the extensions are *classes* of items. In agreement with a long tradition in philosophy, the distinction between specific and individual universals must be admitted to be well-founded; and it is at least plausible if not true that terms occurring in specific universals are in general not associated with any extensions at all, or that if the terms do have extensions the extensions are not classes. But it by no means *follows* from these premises that it is an error to modify common usage by *stipulating* that in certain contexts classes are to be the extensions of such terms. If it is an error nonetheless, Kaufmann has not shown that it is. On the other hand, if it is not an error, the set-theoretical definition of cardinals is a viable alternative to his definition of them in terms of the ordinals. In that case, the question which mode of definition is the preferred one can then have no *a priori* answer, and can be decided only after ascertaining the relative merits of the two modes of definition in making it possible to attain specific objectives. It is conceivable, for example, that the ordinal definition is better suited for performing one task (e.g., clarifying the nature of counting and the logic of measurement), while the set-theoretical definition is more useful in undertaking another (e.g., providing a set-theoretical foundation for a comprehensive systematization of the various branches of mathematics).

Kaufmann's argument for his contention that the idea of non-denumerable infinities is absurd, is also inconclusive for the reasons just stated, so that nothing further need be said about it. Moreover, although he used the important notion of *constructive* definitions and proofs, he used it in an informal, intuitive manner, without stating precisely just what is the distinction between constructive and non-constructive definitions and proofs in mathematics. Indeed, the distinction was not clearly formulated until the theory of recursive functions was developed after the publication of this book. Kaufmann also believed that the so-called 'second order' (or 'higher') logical calculus – which deals with statements ascribing properties (or attributes) to properties – is not needed in general, and in particular not in mathematics. For example, the statement 'The relation of being greater in magnitude is asymmetrical' is a second-order statement, since it ascribes the property of being asymmetrical to the relational property of being greater in magnitude. But this second-order statement is eliminable, for its content is fully rendered by the first-order statement 'If one "object" (e.g., a number) is greater in magnitude than a second, then the second is not greater in magnitude than the first'. However, although many second-order attributions are eliminable because they are logically equivalent in content to first-order statements, this cannot be done always. This becomes evident in defining the notion of one number being *a successor* of another in a sequence of numbers generated by the relation of one number being *the immediate successor* of another. The required definition can be stated as follows: '$y$ is a successor of $x$' if and only if 'There is a class of numbers $C$ of which $y$ is a member but $x$ is not, and every number $z$ belonging to $C$ is either the immediate successor of $x$ or is the immediate successor of some number in $C$'. It is clear that the definition makes mention of a certain class $C$ which is described in terms of its members and has the described property ascribed to it; and since the definition contains the class term $C$ existentially quantified, the term is not eliminable.

Some of these critical comments on Kaufmann's claims are doubtless debatable. However, it is no longer a matter for serious debate whether his account of the structure of counting also established, as he believed, the consistency as well as the completeness of Peano's axioms. For the untenability of this belief became evident with the appearance in 1931 of the Gödel paper to which reference has already been made. Kaufmann's book was published a year earlier, and his claim concerning the consistency and completeness of arithmetic was not wholly unwarranted at the time it was made. Although a number of the views presented in this book must be corrected in the light of later developments in the subject, the book was never revised; nor did Kaufmann leave any indications of what changes in his philosophy of mathematics he thought were made necessary by Gödel's discoveries. But despite these limitations, his book remains an enlightening and stimulating contribution to a fundamental branch of philosophical inquiry.

ERNEST NAGEL

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# EDITOR'S NOTE

Felix Kaufmann (1895–1949) represented, in the way described in Ernest Nagel's introduction, the intersection of the Vienna Circle and the phenomenological movement. His thinking may fairly be said to combine the merits of the two schools. We publish here his main writings in logic and mathematics. Chief of these is the work on the infinite (its author's favourite book): grateful acknowledgement for permission to publish a translation of this must go to Franz Deuticke of Vienna, the house (happily still flourishing) which first published it in 1930. The book was reprinted in German by the Wissenschaftliche Buchgesellschaft of Darmstadt in 1968. There follows an article from *Erkenntnis* **2** (1931), for permission to publish a translation of which we are indebted to the house of Felix Meiner. An early version of this article found among Kaufmann's papers is marked 'Schlick Kreis 13.XI.1930' and was no doubt delivered as a lecture on that day. Finally we have included an unpublished paper of about 1931. This was kindly supplied by the Centre for Advanced Research in Phenomenology at Wilfrid Laurier University, Waterloo, Ontario, where Professor José Huertas-Jourda and Dr. Harry P. Reeder were in every way most helpful. Dr. Reeder has put us further in his debt by a bibliography of Kaufmann's publications, designed for the present volume.

As the editor responsible in this case I am particularly indebted to Dr. Else Kaufmann, the author's widow, who greeted visits and enquiries with encouragement rather than patience. She and their son, Mr. George Kaufmann, have helped to preserve Kaufmann's work and have agreed most readily and on most generous terms to its publication when that was urged upon them. Happy in his heirs, in name, in nature, and only not in length of life, Felix Kaufmann seems to us to merit study for the variety of his gifts and for the particular turn he gave to the ideas of the Vienna Circle. Two further volumes of his writings are planned.

B. McGUINNESS

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