CERTIFICATE MATHEMATICS

C V DURELL

WITH ANSWERS

VOLUME 1

CERTIFICATE MATHEMATICS

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CERTIFICATE MATHEMATICS

VOLUME I

PREFACE

Certificate Mathematics is planned to meet the requirements of the 'General Certificate of Education' *at ordinary level*, as specified by the syllabuses of the various examining bodies. It is arranged in four volumes, each of which forms a year's work for the four years from 11 plus to 15 plus.

Certificate Mathematics is derived from the author's original *General Mathematics*. There are, however, especially in the later volumes, substantial changes of order, treatment, and subject-matter, designed to meet the special purpose of this course; advantage has also been taken of suggestions made by teachers. Fresh material has been introduced in order to meet all the requirements of the relevant examination syllabuses, while some topics and processes not required in the examinations have been excluded.

Attention may be called to the following features:

(i) There is frequent revision to secure familiarity with basic ideas, useful facts and standard processes. Such work is given at appropriate places in the text and is emphasised by the sets of 'Quick Revision' papers at the end of each volume. These papers should form a valuable preparation for examinations, particularly if access to previous volumes is impracticable.

(ii) The primary object of a homogeneous course is to encourage the pupil to select on every occasion the most appropriate method, whether algebraic or geometrical or graphical or trigonometrical. The text and exercises are planned with this end in view. At the same time, great care has been taken to secure that the explanation of a new process is followed by *straightforward* examples which enforce the method without the distraction of miscellaneous applications. In cases where the drill element is of the first importance, practice is given in examples of a single type before another type is introduced, miscellaneous types being included later in the exercise; this applies to all subjects alike.

(iii) A number of exercises are headed 'oral' or 'class discussion'; these are constructed so that much of the working can be done mentally and are designed to accustom the pupil to the *routine of the argument*. It is suggested that these exercises should be taken *viva voce*, but that all pupils should be required to write down the answers.

The examples in the other exercises are classified under two main heads:

(a) A first course: plain numbers.
(b) A parallel course: numbers enclosed in brackets.

The parallel course provides extra practice, if needed; it does not extend the ground covered in the first course. Some examples marked with an asterisk are intended for those pupils who run ahead of the class.

(iv) The geometry in Volume I is restricted to the work of Stage A because experience has shown that progress in the work of Stage B is far more rapid if the pupil is thoroughly familiar with the basic ideas and the fundamental facts.

This is secured by extensive training in the use of instruments. The standard of accuracy in drawing demanded by examining bodies, although properly far below that of the professional draftsman, can only be attained by frequent practice; such work is used to investigate and illustrate the group of properties on which Stage B geometry is based.

(v) The customary practice of numbering theorems has been discontinued. This removes the risk of a pupil referring to a theorem by a number and forces him to give reasons in verbal form; an 'abbreviation for reference' is suggested for each theorem to help him to do so.

(vi) Comprehensive *Tests in Computation* and sets of miscellaneous *Revision Papers* are given at the end of each volume.

The author acknowledges with thanks permission to include questions set in the examinations for the *General Certificate of Education* by the following bodies: Cambridge Local Examinations Syndicate, Senate of the University of London, Joint Matriculation Board of the Northern Universities, Delegates of the Oxford Local Examinations, Oxford and Cambridge Schools Examinations Board, Welsh Joint Education Committee. Such questions are marked C, L, N, O, OC, W in the text.

NOTE ON NEW EDITION

In this new edition the currency has been decimalised, and SI metric units of weight and measure, with their proper abbreviations, have been used throughout.

The essential characteristics of the order and treatment of *Certificate Mathematics* have been preserved, but a few extra topics have been added which are now generally accepted as belonging to the traditional course. Volume I contains a chapter on Number Systems; Volume II includes a fuller treatment of the linear function, half planes and linear inequalities; Volume III includes the trigonometry of angles of any magnitude, an introduction to three-dimensional coordinates, and quadratic inequalities; Volume IV contains a section on arithmetical and geometrical progressions.

Certificate Mathematics was originally available in two forms. Volumes I and II were common to both courses, but Volumes IIIA, IVA were intended for the traditional syllabus, and Volumes IIIB, IVB for the alternative syllabus. Since the first of these has largely disappeared, and has been replaced by what was at first the alternative syllabus, there is no longer any need for two courses, and *Certificate Mathematics* is now published in four volumes.

TABLES

LENGTH

10 millimetres (mm)	= 1 centimetre (cm)
100 centimetres	= 1 metre (m)
1000 metres	= 1 kilometre (km)

CAPACITY

1000 millilitres (ml) = 1 litre (l) = 1000 cm^3

WEIGHT

1000 milligrammes (mg)	= 1 gramme (g)
1000 grammes	= 1 kilogramme (kg)
1000 kilogrammes	= 1 tonne (t)

MONEY

British

100 new pence (p) = 1 pound (£)

American

100 cents = 1 dollar ($)

French

100 centimes (c) = 1 franc (fr)

TIME

60 seconds (s)	= 1 minute (min)
60 minutes	= 1 hour (h)
24 hours	= 1 day
7 days	= 1 week
365 days	= 1 ordinary year
366 days	= 1 leap year
100 years	= 1 century

ANGLE

60 seconds (60″)	= 1 minute (1′)
60 minutes	= 1 degree (1°)
90 degrees	= 1 right angle

CONTENTS

CHAPTER 1

FIRST NOTIONS OF ALGEBRA

Unknown Numbers It is often convenient to use a letter to represent a number whose value is not known at the start but which we expect to be able to discover by making use of information we possess. In this case, the letter is simply the name of what is called the *unknown* number whose value can be obtained, sometimes easily and sometimes only by a difficult argument.

EXERCISE 1 (Oral)

1 What number does x represent in Fig. 1, if AB is 10 cm long?
2 What number does x represent in Fig. 1, if AB is 12 cm long?
3 Find the length of AB in Fig. 1, if x represents the number 2.

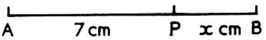

FIG. 1

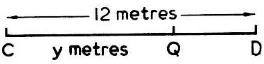

FIG. 2

4 What number does y represent in Fig. 2, if QD is 4 m long?
5 What number does y represent in Fig. 2, if QD is 3 m long?
6 Find the length of QD in Fig. 2, if y represents the number 7.
7 Fig. 3 represents the numbers 1, 2, 3, . . . , 9 arranged in a 'magic square'; the three numbers in each row, in each column, and along each of the two *diagonals*, add up to 15. Sketch the framework and insert in it the given numbers; then fill in the numbers which the letters *a, b, c, d, e, f* represent.

A line which is drawn from one corner of a figure to an opposite corner is called a **diagonal** of the figure.

4	a	2
d	b	e
c	f	6

FIG. 3

8 Fig. 4 represents the numbers 1 to 16 arranged in a 'magic square'; the four numbers in each row, in each column, and along each of the two diagonals, add up to 34. Sketch the framework and insert in it the given numbers; then fill in the numbers which the letters represent. [Start with the diagonals.]

1	a	b	12
c	4	d	6
e	5	16	f
8	g	2	h

FIG. 4

9 I leave home each morning at t minutes past nine. What number does t represent if I leave home at (i) a quarter past nine, (ii) half past nine, (iii) twenty minutes to ten?

10 I stop work at k minutes past four. What time is this if k represents the number 50?

11 A man was n years old on March 1, 1950. What number does n represent if the man (i) was born on March 1, 1920, (ii) will be 70 on March 1, 1980?

12 Fig. 5 shows a plot of ground 20 m by 15 m, with a path t metres wide all round. All lengths are in metres.

(i) What numbers do l, b represent if t represents the number 2?

(ii) What numbers do t, b represent if l represents the number 14?

(iii) What numbers do t, l represent if b represents the number 13?

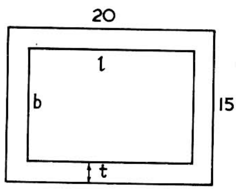

FIG. 5

Symbols The generalisation of Arithmetic, where letters are used to represent numbers, is called **Algebra.**

The symbols +, −, ×, ÷ have the same meanings in Algebra as in Arithmetic. The following symbols are in common use:

$=$ means 'is equal to'; thus $5-2 = 3$.
$\therefore$ means 'therefore'.

Do not confuse $=$ **with** $\therefore$; use the symbol, $=$, as a *verb*. For example, 1 metre $=$ 100 centimetres, $\therefore$ 4 metres $=$ 400 centimetres.

Factors and Multiples Since $42 = 6\times 7$, 6 and 7 are called **factors** of 42, and 42 is called a **multiple** of 6, also a multiple of 7.

$$42 = 2\times 3\times 7, \text{ but it is shorter to write } 42 = 2 \,.\, 3 \,.\, 7,$$

where the dots are placed *at the foot of the line.*

Similarly, $2\times a\times b$ may be written $2\,.\,a\,.\,b$; but it is usually written in the shorter form $2ab$, where the dots are omitted.

The dots in the *product* 2 . 3 . 7 cannot be omitted because 237 means 2 hundreds + 3 tens + 7; but when numbers are represented by letters

ab always means $\mathbf{a}\times\mathbf{b}$.

Just as $1\times 6 = 6\times 1 = 6$, so, whatever number a represents,

$1a = 1\times a = a\times 1 = a$; we therefore write a instead of $1a$.

A phrase, such as 'a represents the number 7', is usually written more shortly in the form, $a = 7$.

The numbers 1, 2, 3, 4, 5, 6, . . . are called **whole numbers** or **integers.** A whole number of which 2 is a factor is called **even;** whole numbers which are not even are called **odd.** Thus 2, 4, 6, 8, . . . are even and 1, 3, 5, 7, . . . are odd.

EXERCISE 2 (Oral)

Express as the product of *two* factors in as many ways as possible:

1 12 **2** 18 **3** 28 **4** 30 **5** 36 **6** 96 **7** 108

Complete the following:

8 $63 = 7 \times \ldots$ **9** $72 = 8 \times \ldots$ **10** $75 = 5 \times \ldots$ **11** $132 = 11 \times \ldots$

12 $36 = 2 \times \ldots = 2 \times 2 \times \ldots$ **13** $84 = 7 \times \ldots = 7 \times 3 \times \ldots$

14 $180 = 10 \times \ldots = 10 \times 9 \times \ldots$ **15** $120 = 8 \times \ldots = 8 \times 3 \times \ldots$

16 $154 = 2 \times \ldots = 2 \times 7 \times \ldots$ **17** $216 = 6 \times \ldots = 6 \times 6 \times \ldots$

18 $12a = 4 \times \ldots$ **19** $30b = 3b \times \ldots$ **20** $15cd = 5c \times \ldots$

21 $9ef = 3e \times \ldots$ **22** $6mn = 2 \times 3m \times \ldots$ **23** $16xy = 2 \times 2y \times \ldots$

24 $3a \times 3 = \ldots$ **25** $4 \times 4b = \ldots$ **26** $5c \times 5d = \ldots$

Prime Numbers A whole number is called **prime** if it is divisible only by itself and 1.

Thus 2, 3, 5, 7, 11, 13, 17, . . . are prime numbers.

Any whole number which is not prime can be expressed as the product of two or more prime numbers, that is in **prime factors.**

For example, $600 = 6 \,.\, 10 \,.\, 10 = (2 \times 3) \,.\, (2 \times 5) \,.\, (2 \times 5)$
$= 2 \,.\, 2 \,.\, 2 \,.\, 3 \,.\, 5 \,.\, 5.$

Index Notation For brevity,

$2 \,.\, 2$ is written 2^2, called '2 squared' or 'the square of 2';
$2 \,.\, 2 \,.\, 2$ is written 2^3, called '2 cubed' or 'the cube of 2';
$2 \,.\, 2 \,.\, 2 \,.\, 2$ is written 2^4, called '2 to the power 4'; it is also called 'the fourth power of 2'; and so on.

Similarly for other numbers:

$5 \,.\, 5 \,.\, 5 \,.\, 5 \,.\, 5 \,.\, 5$ is written 5^6, called '5 to the power 6'; and so on.

In the symbol 5^6, 6 is called the **index** (*plural* indices); it denotes the number of factors in the product $5 \,.\, 5 \,.\, 5 \,.\, 5 \,.\, 5 \,.\, 5$.

For example, using the index notation,

$$600 = 2 \,.\, 2 \,.\, 2 \,.\, 3 \,.\, 5 \,.\, 5 = 2^3 \,.\, 3 \,.\, 5^2.$$

The index notation is used with letters when they represent numbers:

$a \,.\, a$ is written a^2, called 'a squared';
$a \,.\, a \,.\, a$ is written a^3, called 'a cubed';
$a \,.\, a \,.\, a \,.\, a$ is written a^4, called 'a to the power 4'; and so on.

Similarly, just as $2 \,.\, 2 \,.\, 2 \,.\, 3 \,.\, 5 \,.\, 5$ is written $2^3 \,.\, 3 \,.\, 5^2$, so $a \,.\, a \,.\, a \,.\, b \,.\, c \,.\, c$ is written $a^3 \,.\, b \,.\, c^2$ or for short a^3bc^2, and $3 \,.\, 5 \,.\, x \,.\, x \,.\, y \,.\, y \,.\, y \,.\, y \,.\, z$ is written $15 \,.\, x^2 \,.\, y^4 \,.\, z$ or for short $15x^2y^4z$. In particular, 7^1 means 7, and x^1 means x.

When expressing a number in prime factors, it is best at first to proceed systematically. Take the prime numbers in ascending order 2, 3, 5, 7, 11, . . . and divide as often as possible by any one before passing on to the next.

Example 1 Express 5148 in prime factors.

2	5148
2	2574
3	1287
3	429
11	143
	13

$$\therefore\ 5148 = 2\times 2\times 3\times 3\times 11\times 13$$
$$= 2^2\times 3^2\times 11\times 13$$
$$= \mathbf{2^2 . 3^2 . 11 . 13.}$$

EXERCISE 3

State which of the following numbers are prime, and express the others in prime factors, using the index notation:

1 24	**2** 32	[**3**] 35	**4** 37	[**5**] 75
6 125	[**7**] 43	**8** 111	[**9**] 104	***10** 139
***11** 182	**12** 198	**13** 101	[**14**] 207	***15** 294
[**16**] 300	**17** 315	[**18**] 438	[**19**] 450	**20** 528
***21** 588	[**22**] 648	***23** 693	***24** 754	**25** 868
[**26**] 1111	***27** 3432	***28** 3528	***29** 3773	***30** 6032

Products, Quotients, Powers and Roots

Example 2 Multiply x^2 by x^3.

$$x^2 = x\,.\,x,\ 2\text{ factors};\ x^3 = x\,.\,x\,.\,x,\ 3\text{ factors};$$
$$\therefore\ x^2\times x^3 = x\,.\,x\,.\,x\,.\,x\,.\,x,\ (2+3)\text{ factors},\ = \boldsymbol{x^5}.$$

The method used in Example 2 shows that:

If p and q are any whole numbers,

$$\boldsymbol{x^p\times x^q = x^{p+q}}.$$

This result is sometimes stated in the form:

In multiplication, **add** *the indices;*

but this applies only to **powers of the same number.**

Example 3 Divide x^5 by x^3.

$$x^5 = x\,.\,x\,.\,x\,.\,x\,.\,x,\ 5\text{ factors};$$

$\therefore$ by repeated division by x, just as in Example 1,

$$x^5\div x = x\,.\,x\,.\,x\,.\,x,\ 4\text{ factors},\ = x^4,$$
$$x^4\div x = x\,.\,x\,.\,x,\ 3\text{ factors},\ = x^3,$$
$$x^3\div x = x\,.\,x,\ 2\text{ factors},\ = x^2;$$
$$\therefore\ x^5\div x^3 = x\,.\,x,\ (5-3)\text{ factors},\ = \boldsymbol{x^2}.$$

The method used in Example 3 shows that:

If p and q are whole numbers such that p is greater than q,

$$\boldsymbol{x^p\div x^q = x^{p-q}}.$$

This result is sometimes stated in the form:

In division, **subtract** *the indices;*

but this applies only to **powers of the same number.**

At first, letters were used to represent numbers whose values could be discovered from information supplied. In Examples 2, 3, general statements or rules have been set down by using letters to represent any numbers which satisfy the given conditions. In order to understand the meaning of a rule, it is essential to *think in terms of numbers when using letters.*

Example 4 Write in index notation

(i) the square of $2^3 . 5^2$, (ii) the cube of $3^2 . 7$.

(i) $(2^3 . 5^2)^2 = (2^3 . 5^2).(2^3 . 5^2) = 2^3 . 2^3 . 5^2 . 5^2 = \mathbf{2^6 . 5^4}$.

(ii) $(3^2 . 7)^3 = (3^2 . 7).(3^2 . 7).(3^2 . 7) = 3^2 . 3^2 . 3^2 . 7 . 7 . 7 = \mathbf{3^6 . 7^3}$.

Notice that the square of a number is obtained by doubling the index of each factor and that the cube is obtained by multiplying the index of each factor by 3. 7 is the same as 7^1; 3 times the index 1 gives the index 3.

Example 5 Find (i) the square, (ii) the cube, of a^4.

(i) $(a^4)^2 = a^4 . a^4 = a^{4+4} = \boldsymbol{a^8}$.

(ii) $(a^4)^3 = a^4 . a^4 . a^4 = a^{4+4} . a^4 = a^{4+4+4} = \boldsymbol{a^{12}}$.

The **square root** of a given number is the number whose square is equal to the given number.

For example, the square root of 25 is 5 because $5^2 = 25$. If a number has an *exact* square root, it is called a *perfect square.*

For example, 9 is a perfect square, namely the square of 3; but 6 is not a perfect square because there is no number whose square is 6.

The *square root* of a number x is denoted by $\sqrt{x}$.

The **cube root** of a given number is the number whose cube is equal to the given number.

For example, the cube root of 64 is 4 because $4^3 = 4 . 4 . 4 = 64$.

The *cube root* of a number x is denoted by $\sqrt[3]{x}$.

Example 6 Express in index notation

(i) the square root of $3^6 . 5^2 . 7^4$, (ii) the cube root of $2^9 . 3^6 . 5^3$.

(i) $(3^3 . 5 . 7^2).(3^3 . 5 . 7^2) = 3^6 . 5^2 . 7^4$;

$\therefore \sqrt{(3^6 . 5^2 . 7^4)} = \mathbf{3^3 . 5 . 7^2}$.

(ii) $(2^3 . 3^2 . 5).(2^3 . 3^2 . 5).(2^3 . 3^2 . 5) = 2^9 . 3^6 . 5^3$,

$\therefore \sqrt[3]{(2^9 . 3^6 . 5^3)} = \mathbf{2^3 . 3^2 . 5}$.

When a number is expressed in *prime* factors, the index of each factor is *even* if the number is a *perfect square*, and the argument used in Example 6 shows that the square root of the number is obtained by halving the index of each prime factor; similarly, the index of each prime factor is divisible by 3 if the number is a perfect cube, and the cube root of the number is obtained by dividing the index of each prime factor by 3.

Example 7 Find (i) the square root, (ii) the cube root, of b^{12}.

(i) $b^6 \times b^6 = b^{12}$; $\therefore \sqrt{b^{12}} = \boldsymbol{b^6}$.
(ii) $b^4 \times b^4 \times b^4 = b^{12}$; $\therefore \sqrt[3]{b^{12}} = \boldsymbol{b^4}$.

When finding the square root of a number by factors, it saves time to choose only factors which are themselves perfect squares.

Example 8 Find by factors the square root of 28 224.

$$28\,224 = 4\,.\,7056 = 4\,.\,4\,.\,1764 = 4\,.\,4\,.\,4\,.\,441 = 4\,.\,4\,.\,4\,.\,9\,.\,49,$$
$$\therefore \sqrt{28\,224} = 2\,.\,2\,.\,2\,.\,3\,.\,7 = 2^3\,.\,3\,.\,7 = \mathbf{168.}$$

EXERCISE 4

Express in prime factors as shortly as possible:

1 4^3 **2** 12^4 **3** $2^3\,.\,6^2$ **4** $8\,.\,12\,.\,15$ **5** $10^2\,.\,30$ **6** $(2^2\,.\,3^3)\times(2^4\,.\,3)$
[7] $(2^3\,.\,5^2)\times(2^3\,.\,5)$ **8** $(10^2\,.\,12)\times(5\,.\,18)$ **9** $(3^6\,.\,5^3)\div(3^2\,.\,5)$
[**10**] $(2^6\,.\,3^2)\div(2^2\,.\,3^2)$ **11** $(4^5\,.\,6^2)\div(18\,.\,16)$

12 The square of (i) $5^3\,.\,7$, (ii) $15\,.\,20$, (iii) $24\,.\,30\,.\,36$

13 The square root of (i) $3^6\,.\,7^4$, (ii) $3^5\,.\,12^3$, (iii) $2^{16}\,.\,9^3$

14 The cube root of (i) $2^6\,.\,5^9$, (ii) $2^2\,.\,6^2\,.\,12^4$, (iii) $16\,.\,24\,.\,36$

Use factors to find the square roots of the following:

15 196 [**16**] 1764 **17** 3969 [**18**] 9064 **19** 12 544

Use factors to find the cube roots of the following:

20 2744 [**21**] 21 952 **22** 91 125 [**23**] 85 184

Simplify the expressions in Nos. 24–27, *giving the argument in full*:

24 $a^3 \times a^3$ **25** $b^4 \times b$ **26** $c^6 \div c^2$ **27** $d^5 \div d$

Simplify the following:

28 $a \times a^5$ [**29**] $b^3 \times b^2$ **30** $2c^2 \times 3c^4$ **31** $3d^3 \times 3d^3$ **32** $e^2 \div e^2$
[**33**] $f^6 \div f^3$ **34** $4g^4 \div 2g$ **35** $8h^8 \div 2h^2$ **36** $2m^2 \times 3mn$
[**37**] $5pq^2 \times 5p^2q$ **38** $6x^2y^3 \times 4x^3y^4$ **39** $6a^3b \div 2ab$
[**40**] $12cd^6 \div 3cd^2$ **41** $6e^3f^4g \div 2efg$ **42** $2m^2 \times 3mn \times 4n^2$
[**43**] $p \times 2p^2 \times 3p^3$ **44** $2x \times 3x^2 \div 6x^3$ **45** $\sqrt{(16a^4b^2)}$
[**46**] $\sqrt{(4c^6d^8)}$ **47** $\sqrt[3]{(8e^6f^{12})}$ [**48**] $\sqrt[3]{(27g^9)}$ **49** $(2h)^2 \times (3h)^3 \div (6h^2)^2$
50 $2mn \times (3mp)^2 \div (6m^3np^2)$

51 Find the least whole number by which $2^5\,.\,5^4\,.\,7^3$ must be multiplied to obtain (i) a perfect square, (ii) a perfect cube.