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Topological Rings

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(Continuation of the Notas de Matemática)

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To Susan, Sarah, Michael, and Lawrence

PREFACE

This text brings to the frontiers of much current research in topological rings a reader having an acquaintance with some very basic point-set topology and algebra, which is normally presented in semester courses at the beginning graduate level or even at the advanced undergraduate level. Many results not in the text and many illustrations by example of theorems in the text are included among the exercises, sufficient hints for the solution of which with references to the pertinent literature are offered so that solving them does not become a major research effort for the reader. Within mentioned constraints, a bibliography intended to be complete is given. Expectations of a reader include some familiarity with Hausdorff, metric, compact and locally compact spaces and basic properties of continuous functions, also with groups, rings, fields, vector spaces and modules, and with Zorn's Lemma.

In view of the readers for whom the book is written, the exposition is more detailed than would be necessary for readers who are mature mathematicians. In addition, quite a bit of algebra, both commutative and noncommutative, is included, since many of those readers will need additional background in algebra to understand parts of the text. Obviously, there is considerable overlap with my earlier text, *Topological Fields*, in this series (North-Holland Mathematics Studies 157, Notas de Matématica (126)), since both require a common core of knowledge, but in some instances the presentation here of such material (*e.g.*, the completion of a commutative Hausdorff group) is quite different from that in *Topological Fields*. I deeply regret the omission of all applications of categorical concepts to topological rings. To have included the requisite background for those for whom the book is written would have greatly lengthened an already long book and overbalanced any introduction to the use of categorical concepts in the theory of topological rings that could reasonably be presented.

This seems a natural place to record significant errors thus far discovered in *Topological Fields*, and an Errata correcting such errors is included.

The book is typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX, with the exception of the indices, which are typeset by Latex. I am deeply grateful to Dr. Yun-Liang Yu, systems programmer of the Duke Mathematics Department, who has patiently guided me through the intricacies of $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX and Duke's computer system. When I began the task of typesetting this volume, I remarked to Dr. Yu, a recent arrival from China, that I felt like "an immigrant who has just gotten off the boat and doesn't know a word of English." Thanks to him, I now have a rudimentary grasp of the language.

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15 March 1993

CONTENTS

CHAPTER I

TOPOLOGICAL RINGS AND MODULES

In this introductory chapter we shall define and give examples of topological rings, modules, and groups, show how they may be introduced by specifying the neighborhoods of zero, and present a few basic constructions.

1 Examples of Topological Rings

By a *ring* is meant an associative ring, not necessarily one having a multiplicative identity. A *ring with identity* is a ring possessing a multiplicative identity 1 such that $1 \neq 0$. Thus a *zero ring*, one having only one element, is not a ring with identity. A ring A is *trivial* if $xy = 0$ for all $x, y \in A$. Any commutative group is thus the additive group of a trivial ring. A zero ring is a particularly trivial ring.

We shall denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ the set of natural numbers (including zero), integers, rationals, real numbers, complex numbers, and quaterions respectively. The set of real numbers greater than zero is denoted by $\mathbb{R}_{>0}$, and those greater than or equal to zero by $\mathbb{R}_{\geq 0}$.

If A is a ring, A^* denotes the set of its nonzero elements, and if A is a ring with identity, $A^\times$ denotes the multiplicative group of its invertible elements.

If X and Y are sets, $X \setminus Y$ denotes the relative complement of Y in X, that is, $X \setminus Y = \{x \in X : x \notin Y\}$, and Y^X denotes the set of all functions from X to Y. The cardinality of a set X is denoted by $\operatorname{card}(X)$.

A topological ring is simply a ring furnished with a topology for which its algebraic operations are continuous:

1.1 Definition. *A topology $\mathcal{T}$ on a ring A is a* **ring topology** *and A, furnished with $\mathcal{T}$, is a* **topological ring** *if the following conditions hold:*

(TR 1) $(x, y) \to x + y$ *is continuous from* $A \times A$ *to* A

(TR 2) $x \to -x$ *is continuous from* A *to* A

(TR 3) $(x, y) \to xy$ *is continuous from* $A \times A$ *to* A

where A is given topology $\mathcal{T}$ and $A \times A$ the cartesian product topology determined by $\mathcal{T}$.

A ring topology on a ring A clearly induces a ring topology on any subring of A, and unless the contrary is indicated, we shall assume that a subring of a topological ring is furnished with its induced topology.

Norms furnish examples of topological rings:

1.2 Definition. *A function N from a ring A to $\mathbb{R}_{\geq 0}$ is a* **norm** *if the following conditions hold for all $x, y \in A$:*

(N 1) $$N(0) = 0$$

(N 2) $$N(x+y) \leq N(x) + N(y)$$

(N 3) $$N(-x) = N(x)$$

(N 4) $$N(xy) \leq N(x)N(y)$$

(N 5) $$N(x) = 0 \text{ only if } x = 0.$$

If N is a norm on a ring A, then d, defined by $d(x, y) = N(x - y)$ for all $x, y \in A$, is a metric. Indeed, (N 1) and (N 5) imply that $d(x, y) = 0$ if and only if $x = y$, (N 3) implies that $d(x, y) = d(y, x)$, and (N 2) yields the triangle inequality, since

$$\begin{aligned} d(x, z) &= N(x - z) = N((x - y) + (y - z)) \\ &\leq N(x - y) + N(y - z) = d(x, y) + d(y, z). \end{aligned}$$

If d is a complete metric, we say that N is a *complete norm*.

Often symbols similar to $||..||$ are used to denote norms.

1.3 Theorem. *Let N be a norm on a ring A. The topology given by the metric d defined by N is a ring topology.*

Proof. Let $a, b \in A$. For all $x, y \in A$,

$$\begin{aligned} d(x + y, a + b) &= N((x + y) - (a + b)) = N((x - a) + (y - b)) \\ &\leq N(x - a) + N(y - b) = d(x, a) + d(y, b). \end{aligned}$$

Hence (TR 1) holds. For all $x \in A, d(-x, -a) = N(-x + a) = N(x - a) = d(x, a)$ by (N 3). Hence (TR 2) holds. Finally, for all $x, y \in A$,

$$\begin{aligned} d(xy, ab) &= N((x - a)(y - b) + a(y - b) + (x - a)b) \\ &\leq N(x - a)N(y - b) + N(a)N(y - b) + N(x - a)N(b). \end{aligned}$$

Hence (TR 3) holds. •

1.4 Theorem. *Let N be a norm on a ring A. For all x, $y \in A$,*

$$|N(x) - N(y)| \leq N(x - y),$$

and hence N is a uniformly continuous function from A (for the metric defined by N) to $\mathbb{R}_{\geq 0}$.

Proof. $N(x) = N((x - y) + y) \leq N(x - y) + N(y)$, so $N(x) - N(y) \leq N(x - y)$. Hence also $N(y) - N(x) \leq N(y - x) = N(x - y)$. Therefore $|N(x) - N(y)| \leq N(x - y)$. •

In view of 1.3, we shall say that a topological ring is *normable* if its topology is defined by a norm, and in §14 we shall give criteria for a topological ring to be normable. A *normed ring* is simply a ring furnished with a norm and hence with the topology defined by that norm.

Norms on rings play a substantial role in analysis:

Example 1. Let X be a set, $\mathcal{B}(X)$ the ring of all bounded real-valued (or complex-valued) functions on X (a function f is bounded if $N(f) < +\infty$, where $N(f) = \sup\{|f(x)| : x \in X\}$). The function N just defined is a complete norm on $\mathcal{B}(X)$, so $\mathcal{B}(X)$ and each of its subrings is a topological ring for the topology defined by N. Special cases: (a) The ring of all bounded continuous functions on a topological space X. (b) The ring of all continuous functions f on a locally compact space X which "vanish at infinity," that is, such that for every $\epsilon > 0$ there is a compact subset K (depending on f) of X such that $|f(x)| \leq \epsilon$ for all $x \in X \setminus K$. (c) The ring of all continuous functions on a compact space X. (A topological space X is *compact* if it is Hausdorff and if every collection of open subsets of X whose union is X contains finitely many members whose union is X, and X is *locally compact* if it is Hausdorff and each point of X has a compact neighborhood.)

Example 2. Let A be the ring of all analytic functions on a connected open subset D of $\mathbb{C}$, and let K be an infinite compact subset of D. Then N, defined by $N(f) = \sup\{|f(z)| : z \in K\}$, is an incomplete norm on A (Exercise 1.2).

Example 3. Let D be a bounded connected open subset of $\mathbb{C}$, and let A be the ring of all continuous complex-valued functions on $\overline{D}$ whose restrictions to D are analytic functions. Then N, defined by

$$N(f) = \sup\{|f(z)| : z \in \overline{D} \setminus D\},$$

is a complete norm on A.

Example 4. Let A be the ring of all continuous real-valued functions f on a closed bounded interval $[a, b]$ such that f has a continuous derivative f'

on (a, b), and $\lim_{x \to a+} f'(x)$ and $\lim_{x \to b-} f'(x)$ both exist. Then N, defined by $N(f) = \sup\{|f(x)| : a \leq x \leq b\} + \sup\{|f'(x)| : a < x < b\}$, is a complete norm on A.

Example 5. Let $L^1(\mathbb{N})$ be the set of all sequences $(a_i)_{i \geq 0}$ of real numbers such that $\sum_{i=0}^{\infty} |a_i| < +\infty$, and let N be defined on $L^1(\mathbb{N})$ by

$$N((a_i)_{i \geq 0}) = \sum_{i=0}^{\infty} |a_i|.$$

Addition on $L^1(\mathbb{N})$ is defined by $(a_i)_{i \geq 0} + (b_i)_{i \geq 0} = (a_i + b_i)_{i \geq 0}$. Under either of the following two multiplications $L^1(\mathbb{N})$ is a ring and N is a complete norm on $L^1(\mathbb{N})$: (a) pointwise multiplication, *i.e.*, $(a_i)_{i \geq 0}(b_i)_{i \geq 0} = (a_i b_i)_{i \geq 0}$; (b) convolution, *i.e.*,

$$(a_i)_{i \geq 0} * (b_i)_{i \geq 0} = (\sum_{j=0}^{i} a_j b_{i-j})_{i \geq 0}.$$

For an example of a nonmetrizable (in particular, a nonnormable) topological ring, it suffices to take the cartesian product of uncountably many nonzero topological rings, in view of the following theorem:

1.5 Theorem. *The cartesian product of a family $(A_\lambda)_{\lambda \in L}$ of topological rings is a topological ring.*

We shall prove a more general theorem:

1.6 Theorem. *Let $(A_\lambda)_{\lambda \in L}$ be a family of topological rings, let A be a ring, and let $(f_\lambda)_{\lambda \in L}$ be a family of functions such that for each $\lambda \in L$, f_λ is a homomorphism from A to A_λ. The weakest topology on A for which each f_λ is continuous is then a ring topology.*

Proof. That topology has as a basis of open sets all finite intersections of sets of the form $f_\lambda^{-1}(O_\lambda)$ where $\lambda \in L$ and O_λ is open in A_λ. It follows at once that a function g from a topological space B to A is continuous for this topology if and only if $f_\lambda \circ g$ is continuous from B to A_λ for each $\lambda \in L$. In particular, let $B = A \times A$, and let g be either addition or multiplication on A, g_λ the corresponding composition on A_λ. By the preceding, to show that g is continuous, it suffices to show that $f_\lambda \circ g$ is continuous from $A \times A$ to A_λ for all $\lambda \in L$. But $f_\lambda \circ g = g_\lambda \circ (f_\lambda \times f_\lambda)$, where $f_\lambda \times f_\lambda$ is the function $(x, y) \to (f_\lambda(x), f_\lambda(y))$ from $A \times A$ to $A_\lambda \times A_\lambda$. Since f_λ and g_λ are continuous, so is $g_\lambda \circ (f_\lambda \times f_\lambda)$. Thus g is continuous, and hence the topology is a ring topology. •

Theorem 1.5 thus follows by applying 1.6 to the case where

$$A = \prod_{\mu \in L} A_\mu$$

and for each $\lambda \in L$, $f_\lambda = pr_\lambda$, the canonical projection from $\prod_{\mu \in L} A_\mu$ to A_λ (defined by $pr_\lambda((x_\mu)_{\mu \in L}) = x_\lambda$).

1.7 Corollary. *If $(\mathcal{T}_\lambda)_{\lambda \in L}$ is a family of ring topologies on a ring A, then $\sup_{\lambda \in L} \mathcal{T}_\lambda$ is a ring topology.*

Proof. That topology is the weakest on A such that for each λ, the identity mapping from A to A, furnished with topology $\mathcal{T}_\lambda$, is continuous. •

If $\mathcal{T}_1 \dots \mathcal{T}_p$ are topologies on a ring A defined by norms $N_1, \dots N_p$, it is easy to see that $\sup_{1 \le i \le p} N_i$ is a norm defining the topology $\sup_{1 \le i \le p} \mathcal{T}_i$. This permits us to construct some unusual norms, for example, on the field $\mathbb{C}$ of complex numbers. For this, we first observe that the only continuous automorphisms of $\mathbb{C}$ are the identity automorphism and the conjugation automorphism $z \longrightarrow \bar{z}$. Indeed, if σ is a continuous automorphism of $\mathbb{C}$, then $\sigma(x) = x$ for all $x \in \mathbb{Q}$, the prime subfield of $\mathbb{C}$, so as σ and the identity function must agree on a closed set, $\sigma(x) = x$ for all $x \in \mathbb{R}$. On the other hand, as $\sigma(i)^2 = \sigma(i^2) = \sigma(-1) = -1$, $\sigma(i)$ must be either i or $-i$. It readily follows that σ is the identity automorphism in the former case, the conjugation automorphism in the latter.

By the general theory of algebraically closed fields, however, there are nondenumerably many automorphisms of $\mathbb{C}$, so there exists a noncontinuous automorphism σ . We may further assume, by replacing σ with its composite with the conjugation isomorphism, if necessary, that $\sigma(i) = i$. Let $N(z) = \sup\{|z|, |\sigma(z)|\}$. Then N is a norm inducing the usual absolute value on the subfield $\mathbb{Q}(i)$ of C, but, as we shall see later (Corollary 13.13), the completion of $\mathbb{C}$ for the metric defined by N may be identified with the ring $\mathbb{C} \times \mathbb{C}$ and hence contains proper zero-divisors (*i.e.*, nonzero zero-divisors).

1.8 Definition. *Let K be a division ring. An* **absolute value** *on K is a norm A such that $A(xy) = A(x)A(y)$ for all x, $y \in K$.*

It follows that $A(1) = 1$ since $A(1) = A(1)A(1)$ and $A(1) \neq 0$; more generally, if z is a root of unity, (*i.e.*, if $z^n = 1$ for some $n \geq 1$), then $A(z) = 1$.

The most familiar absolute values, of course, are the usual absolute values on $\mathbb{R}, \mathbb{C}$, $\mathbb{H}$, and their subfields.

If A is an absolute value on a division ring K, the elements x of A satisfying $A(x) < 1$ may be characterized topologically as those elements x

such that $\lim_{n\to\infty} x^n = 0$; in any topological ring, such an element is called a *topological nilpotent*.

For any division ring K, the function A_d, defined by $A_d(0) = 0$ and $A_d(x) = 1$ for all $x \in K^*$, is called the *improper absolute value* since the topology it defines is the discrete topology. Moreover, it is the only absolute value on K defining the discrete topology. Indeed, if A is an absolute value other than A_d, then $A(x) \neq 1$ for some $x \in K^*$, so either x or x^{-1} is a topological nilpotent, and therefore the topology defined by A is not the discrete topology. In particular, the only absolute value on a finite field is the improper absolute value. An absolute value on a division ring is *proper* if it is not the improper absolute value.

1.9 Definition. *Absolute values on a division ring are* **equivalent** *if they define the same topology.*

1.10 Theorem. *Let A_1 and A_2 be proper absolute values on a division ring K. The following statements are equivalent:*

1° *A_1 and A_2 are equivalent.*
2° *The topology defined by A_2 is weaker than that defined by A_1.*
3° *For all $x \in K$, if $A_1(x) < 1$, then $A_2(x) < 1$.*
4° *$A_2 = A_1^r$ for some $r > 0$.*

Proof. If 2° holds and if $A_1(x) < 1$, then x is a topological nilpotent for the topology defined by A_1 and *a fortiori* for the weaker topology defined by A_2, so $A_2(x) < 1$. Assume 3°. As A_1 is proper, there exists $x_0 \in K$ such that $A_1(x_0) > 1$. Then $A_1(x_0^{-1}) < 1$, so $A_2(x_0^{-1}) < 1$, and therefore $A_2(x_0) > 1$. Let

$$r = \log A_2(x_0)/\log A_1(x_0).$$

Let $x \in K^*$, and let $s \in \mathbb{R}$ be such that $A_1(x) = A_1(x_0)^s$. Let $m, n \in \mathbb{Z}$, $n > 0$. If $m/n > s$, then $A_1(x) < A_1(x_0)^{m/n}$, so $A_1(x^n x_0^{-m}) < 1$, thus $A_2(x^n x_0^{-m}) < 1$, and therefore $A_2(x) < A_2(x_0)^{m/n}$. Similarly, if $m/n < s$, then $A_2(x) > A_2(x_0)^{m/n}$. Hence $A_2(x) = A_2(x_0)^s$, so

$$\log A_2(x) = s\log A_2(x_0) = sr\log A_1(x_0) = \log A_1(x_0)^{sr} = \log A_1(x)^r,$$

and therefore $A_2(x) = A_1(x)^r$. •

1.11 Theorem. *Let A be an absolute value on division ring K. The set J of numbers $r > 0$ such that A^r is an absolute value is an interval of $\mathbb{R}_{>0}$ containing $(0, 1]$. Moreover, the following statements are equvalent (where, for any $n \in \mathbb{N}$, $n.1 = 1 + \cdots + 1$ (n terms)):*

1° $J = \mathbb{R}_{>0}$.
2° *For all* $n \in \mathbb{N}$, $A(n.1) \leq 1$.
3° *For all* $x, y \in K$, $A(x+y) \leq \sup\{A(x), A(y)\}$.

Proof. Let $0 < r \leq 1$. For any $c \in (0,1)$, $0 < 1-c < 1$, so $c^r \geq c$ and $(1-c)^r \geq 1-c$, and therefore $c^r + (1-c)^r \geq 1$. Applying this inequality to $c = A(x)/(A(x)+A(y))$ where $x, y \in K^*$, we obtain

$$A(x)^r + A(y)^r \geq (A(x)+A(y))^r \geq A(x+y)^r.$$

Thus $r \in J$. Consequently, if $s \in J$ and $0 < t < s$, then $A^t = (A^s)^{(1/s)t}$, so A^t is an absolute value as $0 < t/s < 1$.

For any absolute value $|..|$, $|n.1| \leq n$ for all $n \in \mathbb{N}$ by induction. Hence if 1° holds, then for all $r > 0$, $A(n.1)^r \leq n$ and hence $A(n.1) \leq n^{1/r}$, so $A(n.1) \leq 1$. Clearly 3° implies 1°.

Assume 2°. As $A(y+z) \leq A(y) + A(z) \leq 2\sup\{A(y), A(z)\}$ for all $y, z \in K$, an inductive argument establishes that for any sequence $(y_i)_{1 \leq i \leq 2^r}$ of 2^r terms,

$$A(y_1 + \cdots + y_{2^r}) \leq 2^r \sup\{A(y_i) : 1 \leq i \leq 2^r\}.$$

Let $x \in K$. Then for any $r \in \mathbb{N}$, if $n = 2^r - 1$,

$$\begin{aligned} A(1+x)^n = A((1+x)^n) &\leq 2^r \sup\{A(\binom{n}{k} x^k) : 0 \leq k \leq n\} \\ &\leq 2^r \sup\{A(x^k) : 0 \leq k \leq n\} = (n+1)\sup\{1, A(x)^n\}, \end{aligned}$$

so $A(1+x) \leq (n+1)^{1/n} \sup\{1, A(x)\}$. Hence $A(1+x) \leq \sup\{1, A(x)\}$. Thus, for any $x, y \in K^*$,

$$\begin{aligned} A(x+y) &= A(x)A(1 + A(x^{-1}y)) \\ &\leq A(x)\sup\{1, A(x^{-1}y)\} = \sup\{A(x), A(y)\}. \bullet \end{aligned}$$

1.12 Definition. *An absolute value A on a division ring K is* **nonarchimedean** *if $A(x+y) \leq \sup\{A(x), A(y)\}$ for all $x, y \in K$,* **archimedean** *if it is not nonarchimedean.*

By 1.11, an absolute value A on a division ring K is archimedean if and only if $A(n.1) > 1$ for some $n \in \mathbb{N}$. Consequently, as a finite field admits only the improper absolute value, a field admitting an archimedean absolute value has characteristic zero.