Measures of Complexity and Chaos

Edited by

Neal B. Abraham

Alfonso M. Albano

Anthony Passamante and

Paul E. Rapp

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Edited by

Neal B. Abraham and
Alfonso M. Albano
Bryn Mawr College
Bryn Mawr, Pennsylvania

Anthony Passamante
Naval Air Development Center
Warminster, Pennsylvania

and

Paul E. Rapp
Medical College of Pennsylvania
Philadelphia, Pennsylvania

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PREFACE

This volume serves as a general introduction to the state of the art of quantitatively characterizing chaotic and turbulent behavior. It is the outgrowth of an international workshop on "Quantitative Measures of Dynamical Complexity and Chaos" held at Bryn Mawr College, June 22-24, 1989. The workshop was co-sponsored by the Naval Air Development Center in Warminster, PA and by the NATO Scientific Affairs Programme through its special program on Chaos and Complexity.

Meetings on this subject have occurred regularly since the NATO workshop held in June 1983 at Haverford College only two kilometers distant from the site of this latest in the series. At that first meeting, organized by J. Gollub and H. Swinney, quantitative tests for nonlinear dynamics and chaotic behavior were debated and promoted [1]. In the six years since, the methods for dimension, entropy and Lyapunov exponent calculations have been applied in many disciplines and the procedures have been refined. Since then it has been necessary to demonstrate quantitatively that a signal is chaotic rather than it being acceptable to observe that "it looks chaotic". Other related meetings have included the Pecos River Ranch meeting in September 1985 of G. Mayer-Kress [2] and the reflective and forward looking gathering near Jerusalem organized by M. Shapiro and I. Procaccia in December 1986 [3].

This meeting was proof that interest in measuring chaotic and turbulent signals is widespread. Those facing limits of precision or length of data sets are hard at work developing new algorithms and refining the accuracy of old ones. Applications to symbolic dynamics and to spatio-temporal dynamics are also now emerging with "complexity" as the byword for what is even a richer subject than "chaos".

The success of the meeting was in large part guaranteed by the enthusiasm of the participants, but without the tireless efforts of a few key persons, the order of the meeting would have fallen victim to the ever looming chaos. Special thanks go to Ann Daudert, secretary of the physics department at Bryn Mawr College, and her assistant, Linath Lin. We also acknowledge the behind-the-scenes and late-night efforts of the staff of the Bryn Mawr Summer Conference Office under the direction of L. Zernicke. Many others of our colleagues and associates contributed as needed, including M.E. Farrell, G. Alman, H. Lin, and N. Tufillaro. To all of them go our warmest gratitude. With help such as theirs, it will always be more of a pleasure than a burden to organize a meeting.

Finally we should acknowledge special efforts that enlivened the meeting. J. Doran and her staff provided excellent meals and refreshments. L. Caruso-Haviland and a small crew of dedicated performers and technical staff enriched one evening with "Chaotic Metamorphoses", an inspired combination of video, cinematography, choreography, and readings. The program notes for the performance are included as part of these

proceedings. Their conference T-shirts, "Complexity and Chaos at Bryn Mawr College", were duly earned.

Perhaps our principal regret (and pleasure) will be the constant task of explaining the scientific meaning of the T-Shirt title in an effort to ride the public relations wave crest.

Neal Abraham
Alfonso Albano
Anthony Passamante
Paul Rapp

August 1989

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COMPLEXITY AND CHAOS

N.B. Abraham, A.M. Albano, and N.B. Tufillaro

Department of Physics
Bryn Mawr College
Bryn Mawr, PA 19010-2899, USA

1. INTRODUCTION

Turbulence was one of the key phenomena that motivated the resurgence of interest in nonlinear dynamical systems. It was, after all, investigations into the mechanisms for turbulence that led Ruelle and Takens to invent the term "strange attractor" in 1971. The turbulence that is described by strange attractors is "turbulence in time" (Schuster, 1988) -- deterministic chaos, or *temporal chaos* in current terminology. In the past decade, a vocabulary for the quantitative characterization of temporal chaos has been developed, and has been used to describe and analyze an incredible variety of phenomena in practically all fields of science and engineering. The dimensions of strange attractors, and the entropies and Lyapunov exponents describing motions on them, have been used to analyze heartbeats and brain waves, chemical reactions, lasers, the economy, x-ray emissions of stars, flames, and fluid flow ...

Yet, this vocabulary is not sufficient to describe turbulence, for turbulence is complexity not only in time but also in space. The vocabulary needs to be enlarged to include the quantitative characterization of spatial complexity and its time evolution. So turbulence, once more, is the motivation for efforts to enlarge the scope of nonlinear dynamics to include a description of *spatio-temporal complexity*. These efforts include the use of modal expansions (reminiscent of the old "mode-mode coupling" theories or "dressed-mode analyses"), spatial correlation functions, coupled lattice models, and new approaches based on the analysis of topological defects.

This workshop was an effort to describe the "state of the field" of the quantitative characterization of complexity and chaos. The presentations at the workshop included in these proceedings go a long way towards that description. To enlarge the context and information for the readers, we append to this introduction a bibliography of some of the most recent references and a selection of other key articles. These are organized both by methodology and the subject to which they are applied. Further references can be found in the various chapters of this volume and in the literature citations of other works listed here. (In the following, papers appearing in this volume are followed by an asterisk).

Measures of Complexity and Chaos
Edited by N.B. Abraham *et al.*
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2. CHARACTERIZING TEMPORAL CHAOS

(a) Calculational techniques, precision, error estimates

Dimensions, entropies and Lyapunov exponents have become the standard measures of temporal chaotic behavior. Of these, perhaps because of the possibility of attaching an intuitive geometric significance to them, and because of the commonly made connection between fractal attractors and chaotic behavior, dimensions have been used most frequently. This is also because the algorithms for their computation seem to be the least cumbersome. However, the relative simplicity of these methods can mask a host of possible errors and difficulties. A comprehensive review of the techniques used and the problems encountered in dimension calculations is given in Theiler's review article (Theiler, 1989).

It is now common for a chaotic time series to be described by a spectrum of dimensions, d_q $(-\infty < q < \infty)$, rather than by just one. The d_q 's are a hierarchy of dimensions introduced by Grassberger (Grassberger, 1983) and by Hentschel and Procaccia (Hentschel and Procaccia, 1983). These are defined by,

$$d_q = \frac{1}{q-1} \lim_{r \to 0} \frac{\log \sum_i p_i^q}{\log r} ,$$

where p_i is the number of points in the i^{th} box of a partition consisting of boxes of size r covering the attractor, typically one that is reconstructed from a time series of a single measured variable by an embedding in an m-dimensional space by the now familiar method of time delays (Packard *et al.*, 1980; Takens, 1980; Mané, 1980). d_0 is the "fractal dimension" or the "capacity" and d_1 is the "information dimension". The "correlation dimension", d_2, is the most widely used member of the hierarchy. Alternatively, the spectrum of scaling indices, $f(\alpha)$, is also used (Halsey *et al.*, 1986), where the variables $(\alpha, f(\alpha))$ are obtained from (q, d_q) by a Legendre transformation:

$$\alpha = \frac{\partial\, [(q-1) d_q]}{\partial\, q} , \qquad f(\alpha) = \alpha q - (q-1) d_q .$$

There exist a number of algorithms for calculating dimensions although that due to Grassberger and Procaccia (Grassberger and Procaccia 1983, 1983a, 1983b) for the correlation dimension remains the most widely used because it remains the easiest to implement. The Grassberger-Procacia algorithm is often called a "fixed volume" technique as it involves counting the average number of points in a hypersphere or hypercube as function of the radius, r. "Fixed mass" algorithms (Termonia and Alexandrovicz, 1983; Badii and Politi, 1985), on the other hand, involve the determination of the radius of a hypersphere that contains a given number of points, N, as a function of N. These numbers scale as r^{d_2} and $N^{\frac{1}{d_1}}$, respectively. The Grassberger-Procaccia correlation integral, $\log\sum p_i^2$, has also been modified into a statistical test for potential forecastibility or hidden recurrent pattterns in an observed time series (Brock and Dechert*).

Though in some cases it appears that dimensions can be reliably extracted with as few as 500 data points, the minimum sufficient number of data points, and the optimum data sampling rate and embedding delay, all depend critically on the uniformity of the strange attractor and its dimension. Sometimes a total number of points as few as $N = 10^d$ for an attractor of dimension d can be sufficient, but various parameters of the embedding need to be optimized. (Note that Smith (Smith, 1988) has a proof that the lower bound of the number of points to avoid spurious results is $N = 42^d$).

There are problems with the standard methods, however, especially when they are used with noise-corrupted data sets of limited size and limited precision (the only kind obtainable from experiments). Each of these lead to different and often overlapping problems. A number of these problems were considered in this workshop.

In constructing an embedding, one must select the sampling rate for the data acquisition of the time series, the total length of the time series, the delay between successive elements of an embedding vector, the spacing in time between the first elements of successive embedding vectors, and the total number of embedding vectors. Arbitrary selection of these parameters can introduce significant systematic errors in the construction or coverage, of the attractor. One key quantity appears to be the window used in the embedding -- i.e., the time spanned by each embedding vector. It is known that the results of dimension calculations depend rather sensitively on this choice. Current lore suggests "rule of thumb" criteria based on the correlation time determined from the inverse of the bandwidth of the signal's power spectrum or the first zero, minimum, or the decay of the envelope of the autocorrelation function. Fraser* and Schuster* suggest improved criteria for making this choice.

Caputo* and Politi* address the problem of data requirements, Lange* investigates systematic errors due to finite precision and noise and proposes strategies for correcting these errors. Kostelich* presents a procedure for decreasing the effects of noise, while Hunt*, Sayers* and Theiler* address the problem of obtaining realistic error estimates for the dimensions and entropies that are extracted.

Another dimension-seeking strategy is to determine an "intrinsic dimension", a minimal embedding dimension for the time series. One procedure is to use a singular value decomposition (Broomhead and King, 1986) to determine the smallest number of orthogonal directions needed to describe the data. When applied locally to get an average local intrinsic dimension, it is known to be fairly robust relative to noise (Passamante *et al.*, 1989). Passamante, Hediger and Farrell* discuss the use of an information theoretic criterion to determine local dimension. Goel and Rao* present other criteria.

Dimensions and additional topological properties of strange sets are also obtainable from unstable periodic orbits. Some aspects of these procedures are discussed by Schuster*, Auerbach*, Glendinning*, Smith* as well as by Gilmore* and Solari*.

Continuously varying signals can also be characterized by smaller sets of information. For example, a Poincaré section of a periodic signal selects the value of the signal once every clock cycle. Externally driven systems have well defined clock frequencies. For autonomous systems Poincaré sections are defined by when the trajectory in the reconstructed attractor crosses a selected hyperplane. Similar data reduction procedures involve the study of the sequence of peak values of the signals or the sequence of time intervals between peaks. The dimension of such a subset is one less than the dimension of the corresponding continuous trajectory.

The logical and more sophisticated extension of these classification schemes involves the invention of a set of symbols used to represent different types of behavior of the system. The sequences of symbols can be analyzed for the "syntax" and "grammar" of the "language" of the dynamics. Conditional probabilities and the relative probability of unique sequences can be used to define a degree of complexity and an entropy production rate for the system. Questions of selection of a set of symbols and enlargement of that set were addressed by Badii*, Crutchfield* and Fraser*, among others.

It has been difficult to calculate Lyapunov exponents from experimental data principally because available procedures are not sufficiently robust against noise in experimental data. Several algorithms have been proposed by a number of investigators (Wolf, 1985; Eckmann *et al.*, 1986; Sato *et al.* 1987; Stoop and Meier, 1988) but others have had their problems implementing these on experimental results. Most experimental data sets are of insufficient precision, sampling rate, or length to permit the use of these algorithms on them with consistent reliability or success. Use of the algorithms on data generated by numerical simulations are obviously more straightforward and have been more universally successful. Glover* presents a technique that calculates Lyapunov exponents more efficiently using the Poincaré sections.

Although entropies K_q can be calculated by means of the same correlation integral that is used to calculate dimensions d_q, they have not been as widely used as dimensions, either. There are growing indications, though, that entropies may be more robust quantities than dimensions (in that they seem to be invariant under linear filtering of the data, while the dimensions are not similarly invariant [see Lange*]) and may well find more use in the future. Though the relationship between power spectra (and thus correlation functions) and entropies is not rigorously established, evidence has been consistently reported that the bandwidth of power spectra and the decay of the envelope of autocorrelation functions are excellent estimators for the entropies. Applications to measures of the complexity of symbolic sequences will increase as ways are explored to reduce the artificial complexity of continuous variables used to describe low dimensional phenomena.

It is also worth noting that internal consistency of the results can be tested since the entropy is a good estimator for the positive Lyapunov exponent (if there is only one). Furthermore, there is the Kaplan-Yorke conjecture (Kaplan, 1979) on the relationship between the dimension and the Lyapunov exponents,

$$d_1 = j + \frac{\sum_{i}^{j} \lambda_i}{|\lambda_{j+1}|}$$

where j is the largest integer for which $0 \leq \lambda_1 + \lambda_2 + \cdots + \lambda_j$. This seems to be particularly accurate for larger systems with only a single positive Lyapunov exponent.

(b) Routes to Chaos and a new standard: Homoclinic and Heteroclinic orbits, Shilnikov chaos

Historically, once some degree of universality was assured, routes to chaos were commonly used as indicators that irregular behavior was indeed dynamical chaos. While these qualitative methods relying on changes in power spectra have fallen in disfavor when more quantitative methods can be applied, the simplicity of a new system following generic routes to chaos is still powerful when one is analyzing experimental data. Nevertheless, countless studies have shown that the universality can break down or become more complex, with truncated period-doubling sequences occuring when there are two controlling parameters rather than one. Quasiperiodicity is equally generic in reaching chaos through locking of the previously incommensurate frequencies and then period doubling of the locked conditions. Only for a few special cases does the incommensurate nature remain until chaos appears with a third frequency.

Another now common route through periodic and chaotic dynamics include sequences of symmetry breaking bifurcations (in systems described by an underlying inversion symmetry) and then glueing bifurcations that restore the symmetry with a higher degree of complexity (Hennequin et al.*).

Chaos related to homoclinic or heteroclinic orbits also seems to come in some relatively standard forms. Signatures include the similarity of the topology of successive trajectories around a topologically simple attractor even though the times for successive trajectories may differ widely. Near these homoclinic orbits there are infinite sets of periodic orbits involving different numbers of spirals. Beyond qualitative measures, it now appears that Poincaré plots of return time maps may be the best indicators of the multileaved structures created by the complex homoclinic chaos. Because of the intricate topological structure of these attractors their sensitivity to noise will also be a subject of considerable ongoing investigation. The structure in the return plots will also probably be a major source of data for studies of symbolic dynamics in the future. The new interest in the dynamics of homoclinic and heteroclinic chaos is driven by the experimental observations of such phenomena in chemical reactions (Argoul* and Arneodo*) and lasers (Arecchi* Arimondo*, Glorieux*, Weiss*).

Homoclinic orbits and coherent transients are now also an ordering feature in the study of spatial dynamics. Nikolaenko* demonstrated the importance of local homoclinic behavior in driving spatio-temporal dynamics and Newell (Newell *et al.*, 1988) has also recently focused on the role of local coherent transients and their propagation as a governing feature of turbulence.

(c) Applications to real-world data

The large variety of fields in which dimensions, entropies, and exponents have been used to characterize complex temporal evolution is an indication of the extent to which these quantities have become elements of a scientific vocabulary that is now practically universal. The worksop saw these quantities used to characterize astrophysical data (Atmanspacher*), dendritic growth (Argoul*), electroencephalographic and electrocardiographic data (Babloyantz*), nerve fibers (Frame*), economics (Brock and Dechert*), epidemics (Schaffer*), fluids (Gollub*, Ciliberto*, Nikolaenko*, Sreenivasan), flames (Sreenivasan) and lasers (Arecchi*, Arimondo*, Glorieux*, Tamm*, Raymer*, Weiss*).

As one of the most basic applications of these methods, dimensions have been used to discriminate between chaos and noise. In many situations, it is possible to distinguish those phenomena that result from the combined effects of extremely many independent processes and which therefore may be regarded as stochastic from those that may be described as low-dimensional deterministic processes. Beyond this, dimension calculations have made possible the direct comparison of computational and experimental results.

An example of a comparison between theory and experiment that goes beyond mere matching of the dimensions of experimental data and results of numerical simulations was presented by Weiss* . He described results obtained with a laser working in a region in parameter space where its operation is described by the Lorenz equations, making it possible to make some very meticulous comparisons between theory and experiment.

(d) From data to dynamics: prediction algorithms

It is almost paradoxical that a chaotic time series, characterized by sensitive dependence to initial conditions which renders its distant future values unpredictable in practice, should be the subject of prediction algorithms (Farmer and Sidorowitz, 1987, 1988; Crutchfield and MacNamara, 1987). Yet, since it is deterministic, it is governed by a dynamical law which is discoverable, at least in principle. Badii and Sepulveda*, Crutchfield*, Mees*, Smith*, and Schuster* present various approaches to the short-term prediction problem.

3. CHARACTERIZING SPATIO-TEMPORAL COMPLEXITY

(a) Modal expansions, spatial correlation functions, coupled lattice models

The characterization of spatially complex nonlinear systems is complicated by the fact that sums of solutions are no longer themselves solutions as in the case of linear systems. The time evolution of nonlinear systems can thus no longer be simply described in terms of independently evolving modes. Nevertheless, the system can still be described by expansions over characteristic spatial modes, but one must recognize that the time evolutions of these modes are coupled.

Another alternative is the use of spatial correlation functions. One definition of turbulence, proposed by Heslot *et al.* (1987), is in terms of the decay of the spatial correlation function in a system with spatio-temporal dynamics which are locally chaotic. However, such a definition will clearly reject as not turbulent systems which are described by a small number of spatial modes with time-dependent amplitudes. Clearly, some intermediate approaches to characterization for systems of moderate size are still needed.

A variety of methods are used by Gollub* to characterize parametrically driven surface waves and by Oppo* to describe lasers with many transverse modes.

Another possibility of describing a spatially inhomogeneous system is to model the system on a lattice in which the dynamics at each lattice point is influenced by interactions with a few near neighbors. Coupled lattice models are discussed by Kapral*, while some problems associated with calculating dimensions for these models are presented by Politi*.

(b) Defect-mediated order-disorder transitions

Defect-mediated turbulence is emerging as a promising paradigm for studying weak turbulence in large aspect ratio systems, i.e., systems in which the size of the basic spatial structure is much less than the size of the system. The inspiration for these ideas comes from analogies with defect-mediated phase transitions in equilibrium systems.

Different theoretical approaches to defect formation in non-equilibrium systems are discussed by Coullet and Procaccia. Coullet* described the usefulness of the Ginzburg-Landau equation in understanding the essential features of defect formation, annihilation, and dynamics. Earlier applications to convective systems were supplemented by illustrations appropriate to large-aperture lasers. In contrast, Procaccia* developed a field theory which described the free dynamics and interdefect forces exhibited at finite range. Aspects of this field theory were illustrated by experimental data from electroconvecting nematics.

Several experimental examples of defect formation in far from equilibrium systems are discussed in Part III of these proceedings. Gollub* discussed the dynamics of parametrically forced surface waves which illustrate temporal chaos at small aspect ratios, and possibly a defect-mediated order-disorder transition at large aspect ratios. Defects might also be found in convective fluid systems such as those described by Ciliberto*, and in lasers with many transverse modes as discussed by Oppo*. The strongest evidence for defect-mediated transitions, however, are to be found in nematics and large aspect-ratio Rayleigh-Bénard convection.