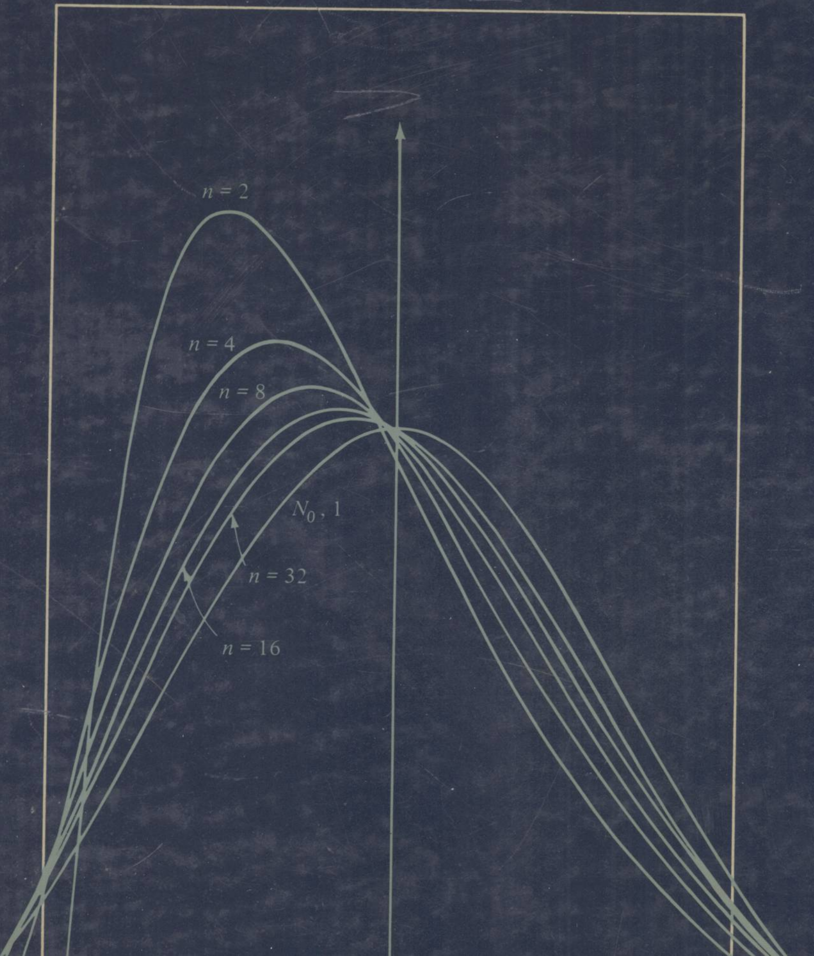
Frederick Solomon

PROBABILITY AND STOCHASTIC PROCESSES



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Frederick Solomon
State University of New York—Purchase

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Probability and Stochastic Processes

To Louise, Benjy, and Alice

". . . and [we ought] always to search for
that which is eternal amidst the random,
everyday events."

Martin Kindder

Preface

This text is designed for a one-semester or two-quarter course in probability theory and applications. Calculus through the techniques of integration is required for the first seven chapters. Chapter 8 requires knowledge of multiple integrals. Although matrix multiplication is used in Chapter 12, Appendix 2 explains the necessary material. In addition, numerous programs are included. These are in a simplified Pascal and only a slight exposure to programming is required.

There is much that is aesthetic in probability. Like any mathematical subject there is an "elegance" that endows the reasoning and the manipulations of symbols with a sense of logical completeness and simplicity. But there is a sense of *magic* when abstract mathematics is seen to apply validly to the "real world." To convey the intuition of the subject has been paramount in writing this book: how reasonable assumptions can be used to formulate a mathematical model which then has applications to the myriad phenomena surrounding us. One does not have to look far to see randomness and chance events. They are everywhere: in the pattern of raindrops, arrival of buses, shapes of natural objects, and so on.

Since probability is an applied subject, I have taken care to include computational ideas. A simplified version of Pascal is used to express algorithms. These are of two types: for calculations of formulas and (beginning with Chapter 7) simulations of random models. Not only is computation necessary in any specific application of probability, but the algorithmic approach provides another means by which we can gain insight into how a model works.

With the exception of Chapter 7, the first ten chapters form the core of a first course in probability. Chapter 7 on simulations can be covered as a single unit or section by section as the topics arise in the earlier chapters; for example, random number generators can be one of the first topics during the first week.

It will not be possible in one semester to cover all the material in Chapters 11 and 12. Neither of these chapters is a prerequisite for the other and can be covered independently. Chapter 11 shows how to solve for the steady state in a birth and death process in continuous time; queues are featured. Chapter 12 is the only one to require matrix multiplication; it covers Markov chains in discrete time.

Throughout the text there are optional sections which contain material at a deeper level of sophistication. If several of these are deleted, a semester course should be able to cover the first ten chapters and at least one of Chapters 11 and 12.

I have tried to convey a sense of the openness of the subject. With simulations particularly, probability affords opportunities for projects and independent work.

I would like to thank Jean Hunter and David Ostrow of Prentice-Hall for their editorial guidance, the reviewers, Professor Galen R. Shorack, the University of Washington, Professor Franklin Sheehan, San Francisco State University, and Professor Donald E. Myers, the University of Arizona, for their excellent suggestions, my family for its patience, and Monique Brion-Escher, Peter Brooks, Jason Choi, Clare Detko, Karl Dushin, Robert Koff, Susan Schroeder, William Widulski, and Robert Drummond—students who made very fine contributions to the quality of the text.

Purchase, New York *Frederick Solomon*

Probability and Stochastic Processes

Contents

The loser, when the game at dice breaks up,
lingers despondent, and repeats the throws to learn,
in grief, what made his fortune droop.

Dante, 1265–1321, The Divine Comedy

Chapter 0

Introduction

Quite naturally we look for relationships and patterns among the variable features in the world around us: We attempt to devise *abstract models*. This means that amidst the infinity of detail in any actual situation we try to strip away the inessential and isolate those variables that are most important to the questions in which we are interested. We do this model building for two reasons: first, for the practicality of it; in applying an abstract model that accurately mirrors an aspect of reality, we are better able to control events; and second, even if a model leads to no practical consequence, understanding the model leads to a sense of insight which is felt to be valuable in its own right.

A mathematical model is one in which the variables and the relationships among them are mathematical. This means that they have a high degree of logical structure: The variables are all numerical or possibly geometric in nature.

It is an oversimplification to say that mathematical model building consists in the observation of variables, the derivation of formulas, and finally the checking of the model's predictions with reality. Rather, the model proceeds through several versions; the checking operation might suggest new variables that are relevant; the derivation of mathematical formulas might suggest new ways in which the model can be checked. The process finally stops when the model is "good enough" for the purpose to which it was intended or one runs out of time or further refinements are too complicated. But the point is that only in the very simplest situations are mathematical models completely successful. There is almost always the possibility of further revision and refinement.

A **random model** is one in which chance and randomness play a significant role. For example, a mathematical model for the amount of traffic on a certain highway cannot possibly assume that all the relevant variables are known; this would imply knowing whether *each* car in the entire city is on the highway. Nor is such detailed knowledge necessary. Rather, a model would make reasonable assumptions and predict the traffic up to likelihoods or various degrees of certainty. How to devise random models, how to interpret them, and how to use them are what this book is about.

Probability deals with the formulation of random models, the derivations of formulas, and the predictions based on them. Mathematical statistics deals with analyzing and revising the model in the light of actual data. For example, a question for probability is: "Given that 40 out of 100 people favor proposition A, what is the likelihood that among 10 people interviewed at random fewer than 3 will be in favor?" A corresponding question for statistics is: "Among 10 people interviewed, 3 favored proposition A. *Given* this, what conclusions can be drawn about the popularity of the proposition among the total population?"

Games of chance are millenia old. Heel bones of hooved animals which have four roughly symmetrical faces were used for the "dice." An Egyptian board game using such a bone dates to 1800 B.C. The casting of lots and other randomizing procedures were common. In the biblical story of Jonah such a device was used to determine that Jonah was responsible for the storm at sea. But there was never more than the most elementary mathematics applied.

As with many other mathematical and scientific fields, the origins of probability lie in sixteenth-century late renaissance. Although the first problems concerned odds in gambling, it is wrong to suppose that the *need* to solve such problems was behind the development of probabilistic methods. Rather, at this time there was a new spirit in the air—a sense of objectivity, an interest in applying logic to phenomena in the natural world, and a faith in our ability to discover scientific-type laws. Gambling games provided the *occasion* for the application of mathematics.

> In this treatise I had in mind foremost the enjoyment of the mathematicians and not the advantages of the players; it is our opinion that those who waste time on games fully deserve to lose their money as well.
>
> *Pierre-Remond de Montmort, 1713*

In the mid-eighteenth century probability was applied to areas other than gambling—first to demography (mortality, census tables, and population studies), then to physics, and theories of random errors of measurement. Presently, all scientific disciplines use methods developed in this text. Precisely because the applications are so varied, the subject itself must be abstract. If a queueing model applies to a waiting line at a telephone booth *as well as* cars waiting at a car wash *as well as* toys on a shelf "waiting" to be sold *as well as* radioactive atoms "waiting" to disintegrate *as well as* playing cards "waiting" to be dealt, . . . , then the model must use an abstract symbolism stripped of specific connotations.

Randomness alone can never produce a significant pattern, for it consists in the absence of any such pattern.

M. Polanyi, Personal Knowledge: Towards a Post-Critical Philosophy.

Chapter 1

The Language and Axioms of Probability

1.1 WHAT IS RANDOMNESS?

In experience there are unpredictable, inexplicable features; the weather is unpredictable, so we say that it is due to randomness as though randomness were a "thing," something to be invoked for lack of a better explanation—for lack of *any* explanation, in fact. A coin is flipped. Will it land heads or will it land tails? If the *exact* force of the thumb on the coin were known and the *exact* position of the fingers and the *exact* air currents and the *exact* composition of the table, *then* one might be able to predict how the coin would land. But these things are not known and to measure them with the required degree of accuracy might well be impossible even in theory. So the *lack* of an explanation for why the coin landed as it did is summarized by the words "randomness" and "chance."

Around the year 1900 the world was felt to be deterministic, by and large. Randomness was an "epiphenomenon"—a category that *we* used to describe the world due to *our* ignorance, but not an essential aspect of the world as it actually is.

> If a minute case which escapes our notice determines a considerable effect which we cannot miss, then we say that this effect is due to chance. If we had an exact knowledge of the laws of nature and the position of the universe at the initial moment, we could predict exactly the position of that same universe in a succeeding moment.
>
> *Henri Poincaré, 1912*