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THEORY OF FINANCIAL DECISION MAKING

Theory of Financial Decision Making

Jonathan E. Ingersoll, Jr.
Yale University

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Preface

In the past twenty years the quantity of new and exciting research in finance has been large, and a sizable body of basic material now lies at the core of our area of study. It is the purpose of this book to present this core in a systematic and thorough fashion. The notes for this book have been the primary text for various doctoral-level courses in financial theory that I have taught over the past eight years at the University of Chicago and Yale University. In all the courses these notes have been supplemented with readings selected from journals. Reading original journal articles is an integral part of learning an academic field, since it serves to introduce the students to the ongoing process of research, including its mis-steps and controversies. In my opinion any program of study would be amiss not to convey this continuing growth.

This book is structured in four parts. The first part, Chapters 1–3, provides an introduction to utility theory, arbitrage, portfolio formation, and efficient markets. Chapter 1 provides some necessary background in microeconomics. Consumer choice is reviewed, and expected utility maximization is introduced. Risk aversion and its measurement are also covered.

Chapter 2 introduces the concept of arbitrage. The absence of arbitrage is one of the most convincing and, therefore, farthest-reaching arguments made in financial economics. Arbitrage reasoning is the basis for the arbitrage pricing theory, one of the leading models purporting to explain the cross-sectional difference in asset returns. Perhaps more important, the absence of arbitrage is the key in the development of the Black-Scholes option pricing model and its various derivatives, which have been used to value a wide variety of claims both in theory and in practice.

Chapter 3 begins the study of single-period portfolio problems. It also introduces the student to the theory of efficient markets: the premise that asset prices fully reflect all information available to the market. The theory of efficient (or rational) markets is one of the cornerstones of modern finance; it permeates almost all current financial research and has found wide acceptance among practitioners, as well.

In the second main section, Chapters 4–9 cover single-period equilibrium models. Chapter 4 covers mean-variance analysis and the capital asset pricing model — a model which has found many supporters and widespread applications. Chapters 5 through 7 expand on Chapter 4. The first two cover generalized measures of risk and additional mutual fund theorems. The latter treats linear factor models and the arbitrage pricing theory, probably the key competitor of the CAPM.

Chapter 8 offers an alternative equilibrium view based on complete markets theory. This theory was originally noted for its elegant treatment of general equilibrium as in the models of Arrow and Debreu and was considered to be primarily of theoretical interest. More recently it and the related concept of spanning have found many practical applications in contingent-claims pricing.

Chapter 9 reviews single-period finance with an overview of how the various models complement one another. It also provides a second view of the efficient markets hypothesis in light of the developed equilibrium models.

Chapter 10, which begins the third main section on multiperiod models, introduces models set in more than one period. It reviews briefly the concept of discounting, with which it is assumed the reader is already acquainted, and reintroduces efficient markets theory in this context. Chapters 11 and 13 examine the multiperiod portfolio problem. Chapter 11 introduces dynamic programming and the induced or derived single-period portfolio problem inherent in the intertemporal problem. After some necessary mathematical background provided in Chapter 12, Chapter 13 tackles the same problem in a continuous-time setting using the mean-variance tools of Chapter 4. Merton's intertemporal capital asset pricing model is derived, and the desire of investors to hedge is examined.

Chapter 14 covers option pricing. Using arbitrage reasoning it develops distribution-free and preference-free restrictions on the valuation of options and other derivative assets. It culminates in the development of the Black-Scholes option pricing model. Chapter 15 summarizes multiperiod models and provides a view of how they complement one another and the single-period models. It also discusses the role of complete markets and spanning in a multiperiod context and develops the consumption-based asset pricing model.

In the final main section, Chapter 16 is a second mathematical interruption—this time to introduce the Ito calculus. Chapter 17 explores advanced topics in option pricing using Ito calculus. Chapter 18 examines the term structure of interest rates using both option techniques and multiperiod portfolio analysis. Chapter 19 considers questions of corporate capital structure. Chapter 19 demonstrates many of the applications of the Black-Scholes model to the pricing of various corporate contracts.

The mathematical prerequisites of this book have been kept as simple as practicable. A knowledge of calculus, probability and statistics, and basic linear algebra is assumed. The Mathematical Introduction collects some required concepts from these areas. Advanced topics in stochastic processes and Ito calculus are developed heuristically, where needed, because they have become so important in finance. Chapter 12 provides an introduction to the stochastic processes used in continuous-time finance. Chapter 16 is an introduction to Ito calculus. Other advanced mathematical topics, such as measure theory, are avoided. This choice, of course, requires that rigor or generality sometimes be sacrificed to intuition and understanding. Major points are always presented verbally as well as mathematically. These presentations are usually accompanied by graphical illustrations and numerical examples.

To emphasize the theoretical framework of finance, many topics have been left uncovered. There is virtually no description of the actual operation of financial markets or of the various institutions that play vital roles. Also missing is a discussion of empirical tests of the various theories. Empirical research in finance is perhaps more extensive than theoretical, and any adequate review would require a complete book itself. The effects of market imperfections are also not treated. In the first place, theoretical results in this area have not yet been fully developed. In addition the predictions of the perfect market models seem to be surprisingly robust despite the necessary simplifying assumptions. In any case an understanding of the workings of perfect markets is obviously a precursor to studying market imperfections.

The material in this book (together with journal supplements) is designed for a full year's study. Shorter courses can also be designed to suit individual tastes and prerequisites. For example, the study of multiperiod models could commence immediately after Chapter 4. Much of the material on option pricing and contingent claims (except for parts of Chapter 18 on the term structure of interest rates) does not depend on the equilibrium models and could be studied immediately after Chapter 3.

This book is a text and not a treatise. To avoid constant interruptions and footnotes, outside references and other citations have been kept to a minimum. An extended chapter-by-chapter bibliography is provided, and my debt to the authors represented there should be obvious to anyone familiar with the development of finance. It is my hope that any student in the area also will come to learn of this indebtedness.

I am also indebted to many colleagues and students who have read, or in some cases taught from, earlier drafts of this book. Their advice, suggestions, and examples have all helped to improve this product, and their continuing requests for the latest revision have encouraged me to make it available in book form.

Jonathan Ingersoll, Jr.

New Haven
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Glossary of Commonly Used Symbols

a	Often the parameter of the exponential utility function $u(Z) = -\exp(-aZ)$.
$\mathbf{B}$	The factor loading matrix in the linear model.
b	Often the parameter of the quadratic utility function $u(Z) = Z - bZ^2/2$.
b_i^k	$= \text{Cov}(\tilde{z}_i, \tilde{Z}_e^k)/(\text{Cov}(\tilde{z}_e^k, \tilde{Z}_e^k)$. A measure of systematic risk for the i^{th} asset with respect to the k^{th} efficient portfolio. Also the loading of the i^{th} asset on the k^{th} factor, the measure of systematic risk in the factor model.
C	Consumption.
E	The expectation operator. Expectations are also often denoted with an overbar $\bar{\ }$.
e	The base for natural logarithms and the exponential function. $e \approx 2.71828$.
$\tilde{f}$	A factor in the linear factor model.
$\mathbf{I}$	The identity matrix.
i	As a subscript it usually denotes the i^{th} asset.
J	A derived utility of wealth function in intertemporal portfolio models.
j	As a subscript it usually denotes the j^{th} asset.
K	The call price on a callable contingent claim.
k	As a subscript or superscript it usually denotes the k^{th} investor.
L	Usually a Lagrangian expression.
m	As a subscript or superscript it usually denotes the market portfolio.
N	The number of assets.
$N(\cdot)$	The cumulative normal distribution function.

$n(\cdot)$ The standard normal density function.

$O(\cdot)$ Asymptotic order symbol. Function is of the same as or smaller order than its argument.

$o(\cdot)$ Asymptotic order symbol. Function is of smaller order than its agrument.

$\mathbf{p}$ The supporting state price vector.

q Usually denotes a probability.

R The riskless return (the interest rate plus one).

r The interest rate. $r \equiv R - 1$.

S In single-period models, the number of states. In intertemporal models, the price of a share of stock.

s As a subscript or superscript it usually denotes state s.

T Some fixed time, often the maturity date of an asset.

t Current time.

$\mathbf{t}$ The tangency portfolio in the mean-variance portfolio problem.

U A utility of consumption function.

u A utility of return function.

V A derived utility function.

$\mathbf{v}$ The values of the assets.

W Wealth.

$W(S, \tau)$ The Black–Scholes call option pricing function on a stock with price S and time to maturity of τ.

$\mathbf{w}$ A vector of portfolio weights. w_i is the fraction of wealth in the i^{th} asset.

X The exercise price for an option.

$\mathbf{Y}$ The state space tableau of payoffs. Y_{si} is the payoff in state s on asset i.

$\mathbf{Z}$ The state space tableau of returns. Z_{si} is the return in state s on asset i.

$\tilde{Z}_{\mathbf{w}}$ The return on portfolio $\mathbf{w}$.

z As a subscript it denotes the zero beta portfolio.

$\tilde{\mathbf{z}}$ The random returns on the assets.

$\bar{\mathbf{z}}$ The expected returns on the assets.

$\mathbf{0}$ A vector or matrix whose elements are 0.

$\mathbf{1}$ A vector whose elements are 1.

$\geq$ As a vector inequality each element of the left-hand vector is greater than the corresponding element of the right-hand vector. $<$ is similarly defined.

$\geqslant$ As a vector inequality each element of the left-hand vector is greater than or equal to the corresponding element of the right-hand vector, and at least one element is strictly greater. $\leqslant$ is similarly defined.

$\geqq$ As a vector inequality each element of the left-hand vector is greater than or equal to the corresponding element of the right-hand vector. $\leqq$ is similarly defined.

α The expected, instantaneous rate of return on an asset.

β $\equiv \text{Cov}(\tilde{z}, \tilde{Z}_m)$. The beta of an asset.

γ Often the parameter of the power utility function $u(Z) = Z^\gamma/\gamma$.

Δ A first difference.

$\tilde{\varepsilon}$ The residual portion of an asset's return.

$\boldsymbol{\eta}$ A portfolio commitment of funds not nomalized.

Θ A martingale pricing measure.

$\boldsymbol{\iota}_j$ The j^{th} column of the identity matrix.

$\tilde{\Lambda}$ The state price per unit probability; a martingale pricing measure.

λ Usually a Lagrange multiplier.

$\boldsymbol{\lambda}$ The factor risk premiums in the APT.

$\mathbf{v}$ A portfolio of Arrow–Debreu securities. v_s is the number of state s securities held.

$\boldsymbol{\pi}$ The vector of state probabilities.

ρ A correlation coefficient.

$\boldsymbol{\Sigma}$ The variance-covariance matrix of returns.

σ A standard deviation, usually of the return on an asset.

τ The time left until maturity of a contract.

Φ Public information.

ϕ_k Private information of investor k.

$\boldsymbol{\omega}$ An arbitrage portfolio commitment of funds ($\mathbf{1}'\boldsymbol{\omega} = 0$).

ω A Gauss–Wiener process. $d\omega$ is the increment to a Gauss–Wiener process.

Contents

List of Tables ix
List of Figures xi
Preface xiii
Glossary of Commonly Used Symbols xvii

Mathematical Introduction 1
Review of notation ... optimization methods ... probability.

1 Utility Theory 19
Utility functions and preference orderings ... Ordinal utility functions ... Consumer demand ... Expected utility maximization ... Cardinal utility ... Utility independence ... Risk aversion ... HARA utility functions ... Multiperiod utility functions.

2 Arbitrage and Pricing: The Basics 45
State space framework ... Redundant assets ... Insurable states ... Dominance and arbitrage ... Supporting prices ... Risk-neutral pricing.

3 The Portfolio Problem 65
Optimal portfolios and pricing ... Properties of portfolios ... Stochastic dominance ... Efficient markets ... Information revelation by prices.

4 Mean-Variance Portfolio Analysis 82
The mean-variance problem ... Covariance properties of minimum variance portfolios ... Expected returns relations ... The Capital Asset Pricing Model ... Consistency with expected utility maximization ... State prices under mean-variance analysis ... Portfolio analysis using higher moments.

Appendix A: The Budget Constraint
Appendix B: Elliptical Distributions

5 Generalized Risk, Portfolio Selection, and Asset Pricing 114
Definition of risk ... Mean preserving spreads ... Rothschild and Stiglitz theorems ... Second-order stochastic dominance ... Opti-

mal and efficient portfolios ... Verifying efficiency ... Risk of securities.

Appendix: Stochastic Dominance

6 Portfolio Separation Theorems 140
Inefficiency of the market portfolio ... One, two and K fund separation under restrictions on utility ... One, two and K fund separation under restrictions on distributions ... Money separation ... Market pricing under separation ... Distinction between pricing and separation.

7 The Linear Factor Model: Arbitrage Pricing Theory 166
Linear factor models ... Residual risk-free models ... Unavoidable risk ... Interpretation of the factor premiums ... Asymptotic arbitrage ... Idiosyncratic risk ... Fully diversified portfolios ... Pricing bounds ... Exact pricing.

8 Equilibrium Models with Complete Markets 186
Valuation ... Portfolio separation ... Pareto optimality ... Effectively complete markets ... Convexity of efficient set ... Creating state securities with options.

9 General Equilibrium Considerations in Asset Pricing 199
Effects of financial contracts ... Systematic and nonsystematic risk ... Market efficiency ... Utility aggregation and the representative investor.

10 Intertemporal Models in Finance 220
State descriptions ... Martingale valuation measures ... Market completion with dynamic trading ... Intertemporally efficient markets ... Infinite horizon models.

11 Discrete-Time Intertemporal Portfolio Selection 235
The intertemporal budget constraint ... Derived utility of wealth ... Hedging behavior

Appendix A: Consumption Portfolio Problem when Utility is Not Additively Separable

Appendix B: Myopic and Turnpike Portfolio Policies

12 An Introduction to the Distributions of Continuous-Time Finance 259
Compact distributions ... Combinations of compact random variables and portfolio selection ... Infinitely divisible distributions.

13 Continuous-Time Portfolio Selection 271
The portfolio problem in continuous time ... Testing the model .... Stochastic opportunity set ... Hedging.

14 The Pricing of Options 298
Restrictions on option prices ... The riskless hedge ... Black-Scholes Model ... Black-Scholes put price ... Preference-free pricing.

15 Review of Multiperiod Models 329
Martingale pricing processes ... Complete markets ... Consumption model ... State-dependent utility ... Restrictions on returns distributions.

16 An Introduction to Stochastic Calculus 347
Diffusion processes ... Ito's lemma ... First passage time ... Maximum and minimum of diffusion processes ... Extreme variation of diffusion processes ... Statistical estimation of diffusion processes.

17 Advanced Topics in Option Pricing 361
Alternate derivation of option model ... Probabilistic interpretation of option equation ... Options with arbitrary payoffs ... Option pricing with dividends ... Options with payoffs at random times ... Perpetual options ... Options with early exercise ... Options with path-dependent values ... Option claims on more than one asset ... Option claims on nonprice variables ... Permitted stochastic processes ... "Doubling" strategies.

18 The Term Structure of Interest Rates 387
The term structure in a certain economy ... Expectations hypothesis ... The term structure in continuous time ... Simple models ... Permissible equilibria ... Liquidity preference and preferred habitats ... Interest rate determinants ... Multiple state variables.

19 Pricing the Capital Structure of the Firm 410
Modigliani-Miller ... Warrants and rights ... Risky bonds ... The risk structure of interest rates ... Cost of capital ... Subordinated debt ... Convertible securities ... Callable bonds ... Sequential exercise of warrants ... Interest rate risk ... Contingent contracting.

Bibliography 449

Index 465

List of Tables

5.1 Summary of Preference Ordering Conditions 123
6.1 Summary of Conditions for Mutual Fund Theorems 164
14.1 Black-Scholes Option Prices 314
14.2 Black-Scholes Option Elasticities 317
14.3 Black-Scholes Option Deltas 318
14.4 Proposition 19 Bounds on Option Prices 326
19.1 Payoffs to Junior and Senior Debt Issues with the Same Maturity 425
19.2 Payoffs to Junior and Senior Debt Issues when Junior Debt Matures First and $B \geqslant b$ 427
19.3 Payoffs to Junior and Senior Debt Issues when Junior Debt Matures First and $b > B$ 427
19.4 Illustration of Optimal Call Policy on Convertible Bonds 432
19.5 Illustration of Monopoly Power in Warrant Holdings 435
19.6 Illustration of Competitive Equilibrium among Warrant Holders 440
19.7 Illustration of Multiple Competitive Equilibria with Warrants 442

List of Figures

1.1	Indifference Curves	22
1.2	Strict Complements	23
1.3	Consumer's Maximization Problem	27
1.4	Income and Substitution Effects	29
1.5	Derived Utility of Wealth Function	36
2.1	State Returns as a Function of w	48
4.1	Mean-Variance Efficient Portfolios	83
4.2	Minimum-Variance Portfolios	84
4.3	Risky-Asset-Only Minimum-Variance Set	85
4.4	Minimum-Variance Set	88
4.5	M-V Problem with Restricted Borrowing and Lending	95
5.1	Mean-Preserving Spread	117
5.2	Original Density Function	118
5.3	"Spread" Density Function	119
5.4	Portfolios with Less Risk than $\tilde{Z}_c$	127
5.5	Portfolios with Higher Expected Returns than $\tilde{Z}_c$	128
5.6	Marginal Utilities in Each State	131
5.7	Marginal Utilities in Each State	133
6.1	Example of Non-Convexity of Efficient Set	142
10.1	History of Dynamically Complete Market	227
10.2	History of Incomplete Market	228
12.1	Example of a Compact Distribution	260
12.2	Example of a Wiener Process	267
12.3	Example of a Poisson Process	268
14.1	Restrictions on Option Values	301
14.2	Black-Scholes Option Values	315
18.1	Illustration of Yield Curves	399
19.1	Yield Spread on Default-Free Bonds	422
19.2	Bond Risk as Percent of Firm Risk	423
19.3	Costs of Capital	424
19.4	Callable and Noncallable Convertibles	433
19.5	Illustration of Sequential Exercise	439

Mathematical Introduction

DEFINITIONS AND NOTATION

Unless otherwise noted, all quantities represent real values. In this book derivatives are often denoted in the usual fashion by $'$, $''$, and so forth. Higher-order derivatives are denoted by $f^{(n)}$ for the n^{th} derivative. Partial derivatives are often denoted by subscripts. For example,

$$\begin{aligned} F_1(x, y) &\equiv F_x(x, y) \equiv \frac{\partial F(x, y)}{\partial x}, \\ F_{12}(x, y) &\equiv F_{xy}(x, y) \equiv \frac{\partial F(x, y)}{\partial x \partial y}. \end{aligned} \tag{1}$$

Closed intervals are denoted by brackets, open intervals by parentheses. For example,

$$\begin{aligned} x \in [a, b] \quad &\text{means} \quad \text{all } x \text{ such that } a \leqslant x \leqslant b, \\ x \in [a, b) \quad &\text{means} \quad \text{all } x \text{ such that } a \leqslant x < b, \end{aligned} \tag{2}$$

The greatest and least values of a set are denoted by Max($\cdot$) and Min($\cdot$), respectively. For example, if $x > y$, then

$$\text{Min}(x, y) = y \qquad \text{and} \qquad \text{Max}(x, y) = x. \tag{3}$$

The relative importance of terms is denoted by the asymptotic order symbols:

$$\begin{aligned} f(x) = o(x^n) \quad &\text{means} \quad \lim_{x \to 0} \frac{f(x)}{x^n} = 0; \\ f(x) = O(x^n) \quad &\text{means} \quad \lim_{x \to 0} \frac{f(x)}{x^{n+\varepsilon}} = 0 \quad \text{for all } \varepsilon > 0. \end{aligned} \tag{4}$$

Dirac delta function

The Dirac delta function $\delta(x)$ is defined by its properties:

$$\delta(x) = \begin{cases} 0, & x \neq 0, \\ \infty, & x = 0, \end{cases} \tag{5}$$

$$\int_{-a}^{a} \delta(x)\,dx = 1 \qquad \text{for any } a > 0.$$

The delta function may be considered as the limit of a mean zero density function as the dispersion goes to zero. For example, for the normal density

$$\delta(x) = \lim_{\sigma \to 0} (2\pi\sigma^2)^{-1/2} \exp\left(\frac{-x^2}{2\sigma^2}\right). \tag{6}$$

In the limit all the probability mass is concentrated at the origin, but the total mass is still unity.

The Dirac delta function is most often used in formal mathematical manipulations. The following property is useful:

$$\int_{a}^{b} \delta(x - x_0) f(x)\,dx = f(x_0) \qquad \text{if } a \leqslant x_0 \leqslant b. \tag{7}$$

Unit step function

The unit step function is the formal integral of the Dirac delta function and is given by

$$u(x) = \begin{cases} 1, & x > 0, \\ \frac{1}{2}, & x = 0, \\ 0, & x < 0. \end{cases} \tag{8}$$

Taylor Series

If f and all its derivatives exist in the region $[x, x + h]$, then

$$f(x + h) = f(x) + f'(x)h + \tfrac{1}{2}f''(x)h^2 + \cdots + \frac{1}{n!}f^{(n)}(x)h^n + \cdots. \tag{9}$$

If f and all its derivatives up to order n exist in the region $[x, x + h]$, then it can be represented by a Taylor series with Lagrange remainder

$$f(x + h) = f(x) + f'(x)h + \cdots + \frac{1}{(n-1)!}f^{(n-1)}(x)h^{n-1} + \frac{1}{n!}f^{(n)}(x^*)h^n, \tag{10}$$

where x^* is in $[x, x + h]$. For a function of two or more arguments the extension is obvious:

$$F(x + h, y + k) = F(x, y) + F_1(x, y)h + F_2(x, y)k$$

$$+ \tfrac{1}{2}F_{11}(x, y)h^2 + \tfrac{1}{2}F_{22}(x, y)k^2 + F_{12}(x, y)hk \qquad (11)$$

$$+ \cdots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x, y) + \cdots.$$

Mean Value Theorem

The mean value theorem is simply the two-term form of the exact Taylor series with Lagrange remainder:

$$f(x + h) = f(x) + f'(x + \alpha h)h \qquad (12)$$

for some α in $[0, 1]$. The mean value theorem is also often stated in integral form. If $f(x)$ is a continuous function in (a, b), then

$$\int_a^b f(x)\,dx = (b - a)f(x^*) \qquad (13)$$

for some x^* in (a, b).

Implicit Function Theorem

Consider all points (x, y) on the curve with $F(x, y) = a$. Along this curve the derivative of y with respect to x is

$$\left.\frac{dy}{dx}\right|_{F=a} = -\frac{\partial F/\partial x}{\partial F/\partial y}. \qquad (14)$$

To see this, note that

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy.$$

Setting $dF = 0$ and solving for dy/dx gives the desired result.

Differentiation of Integrals: Leibniz's Rule

Let $F(x) \equiv \int_{A(x)}^{B(x)} f(x, t)\,dt$ and assume that f and $\partial f/\partial x$ are continuous in t in $[A, B]$ and x in $[a, b]$. Then

$$F'(x) = \int_A^B f_1(x, t)\,dt + f(x, B)B'(x) - f(x, A)A'(x) \qquad (15)$$

for all x in $[a, b]$. If $F(x)$ is defined by an improper integral ($A = -\infty$ and/or $B = \infty$), then Leibniz's rule can be employed if $|f_2(x, t)| \leqslant M(t)$ for all x in $[a, b]$ and all t in $[A, B]$, and the integral $\int M(t)\,dt$ converges in $[A, B]$.