NUMERICAL MATHEMATICS
AND SCIENTIFIC COMPUTATION

Finite Element Methods for Maxwell's Equations

PETER MONK

OXFORD SCIENCE PUBLICATIONS

Finite Element Methods for Maxwell's Equations

Peter Monk
Department of Mathematical Sciences
University of Delaware
Newark, USA

CLARENDON PRESS · OXFORD
2003

OXFORD
UNIVERSITY PRESS

Great Clarendon Street, Oxford OX2 6DP

Oxford University Press is a department of the University of Oxford.
It furthers the University's objective of excellence in research, scholarship,
and education by publishing worldwide in

Oxford New York

Auckland Bangkok Buenos Aires Cape Town Chennai
Dar es Salaam Delhi Hong Kong Istanbul Karachi Kolkata
Kuala Lumpur Madrid Melbourne Mexico City Mumbai Nairobi
São Paulo Shanghai Taipei Tokyo Toronto

Oxford is a registered trade mark of Oxford University Press
in the UK and in certain other countries

Published in the United States
by Oxford University Press Inc., New York

First published 2003

British Library Cataloguing in Publication Data

Data available

Library of Congress Cataloging in Publication Data

ISBN 0 19 850888 3

10 9 8 7 6 5 4 3 2 1

Typeset by the author using LaTeX

Printed in Great Britain
on acid-free paper by
T. J. International Ltd, Padstow

PREFACE

In writing a book on the mathematical foundations of the finite element method for approximating Maxwell's equations I am well aware that I am on very dangerous ground. In his recent textbook *Functional Analysis*, Lax [202] says that "Two souls dwell in the bosom of scattering theory. One is mathematical and handles the unitary equivalence of operators with continuous spectra. The other is physics ...". This quotation seems to me to describe scattering theory remarkably well, except that from the point of view of this book we need to substitute "electrical engineering" for physics. There is currently an enormous effort in the electrical engineering community to simulate electromagnetic phenomena using a variety of numerical methods including finite elements, which are the subject of this book. On the mathematical side there has recently been increased interest in the understanding of the mathematical properties of Maxwell's equations relevant to numerical analysis. The purpose of this book is to describe some of the basic mathematical theory of Maxwell's equations as it pertains to finite element methods, and hence to provide some mathematical underpinnings for the finite element method in this context. Along the way I shall try to point out some of the more obvious problems still remaining. Inevitably, such a book can be criticized on the grounds of being insufficiently mathematical or insufficiently practical (a more likely criticism), depending on the background of the reader — which brings us back to Lax's quotation!

The book is intended to be self-contained from the point of view of finite element theory. Therefore, there is a detailed discussion of convergence theory for mixed finite element methods, basic definitions of finite elements, and error estimates. However, it is much less detailed from the point of view of practical implementation — for this aspect of the finite element method there are already excellent sources in the electrical engineering literature including [177, 272]. Inevitably, it is necessary to assume some mathematics background for the book. Two subjects form the basis of the theory here: functional analysis and Sobolev space theory. For these topics, the excellent book of McLean [215] covers more than is necessary for this book. I have not assumed that the reader is familiar with Sobolev spaces of vector functions. Thus, in Chapter 3, I have summarized some more or less classical material on these spaces. The main source for this chapter is the book of Girault and Raviart [143]. This is a lovely book and well worth reading.

After the preparatory work in Chapters 2 (functional analysis and abstract error estimates) and 3 (Sobolev spaces and vector function spaces) we move on, in Chapter 4, to discuss a simple model problem for Maxwell's equations. This is a cavity or interior problem, which is posed on a bounded domain, but with boundary conditions motivated by scattering applications (as first described in Chapter 1). This chapter uses the spaces from Chapter 3 to write down and analyze a standard variational formulation for the cavity problem. The analysis motivates the function spaces involved and the analytical techniques used to investigate such a problem.

At this stage we face a decision: what class of domains to allow for the scatterer. On the one hand, the theory of partial differential equations is much simplified if the domain has a smooth boundary. But this vastly complicates the discussion of finite element methods and the effects of the approximation of smooth boundaries is not well understood for Maxwell's equations. Therefore, I have decided to focus my discussion on Lipschitz polyhedra. These allow the use of standard tetrahedral meshes. In addition, some of the subtle problems related to approximating Maxwell's equations (such as the non-convergence of standard finite element methods in some cases [105]) appear in this situation. Finally, some of the most interesting recent advances in finite element theory and function space theory for Maxwell's equations has taken place in the context of Lipschitz polyhedral domains (see, e.g. [63, 106, 12]).

Using the discussion of Chapter 4 as motivation, we see that some special finite elements — the edge elements of Nédélec [233] — are particularly well suited to discretizing the Maxwell system. Therefore, in Chapters 5 and 6 we present a detailed description of these spaces, together with an associated scalar space for the electrostatic potential and other spaces needed to complete the theory. These chapters are a central part of the book and, besides presenting the original Nédélec finite element spaces, also emphasize some more recent viewpoints, including in particular the discrete de Rham diagram which summarizes the relationships between the relevant function spaces, their finite element discretizations and interpolation operators.

Having obtained a suitable variational formulation of the cavity problem and suitable finite element spaces, we then move to the finite element discretization of the cavity problem in Chapter 7. I present in detail two proofs of convergence for this method. To date, the first proof can only be applied in a special case, but has the advantages of simplicity and of providing a very clean result. In addition, this theory will be used later when we investigate the frequency dependence of the error in finite element methods in Chapter 13 and when discussing an overlapping Schwarz method for solving the associated matrix problem. The second proof uses the theory of collectively compact operators to prove convergence in a rather general case allowing spatially dependent electromagnetic parameters. Another proof, due to Hiptmair [164], is not included but a similar technique is used later in Chapter 10. A fourth proof, due to Boffi and Gastaldi [50], is also not included since it rests on the theory of eigenvalue problems, which are not an emphasis of this book (although we do provide some theory in Chapters 4 and 7). The three chapters, 4, 5 and 7, form the core of the book and could be useful in a graduate course on finite element methods. Together with some material from Chapter 13 and some from the engineering texts mentioned above, an entire course could be constructed — and indeed this book is partially a result of such a course taught at the University of Delaware. These chapters contain the principal technical results used in all analyses of edge elements to date.

A central task of computational electromagnetism is the approximation of scattering problems. In these problems a known incident field (e.g. from a radar transmitter) interacts with an object (e.g. an aircraft) to create a scattered field. The approximation of this scattered field (or the total field) is the goal of the finite element method. In this book we shall only consider the case of a bounded scatterer (like an aircraft). This reflects my interests, but of course there are many very important applications of scattering from unbounded media. Examples include the classical problem of computing scattering from an infinite periodic structure (or diffraction grating) [25] or a periodic structure with defects [10]. Although we shall not be handling these problems here, the techniques presented also appear in the analysis of more complex problems. For example the theory of Chapter 10 has been used in the analysis of scattering from objects coated by thin layers [11]. Our presentation of scattering problems starts with classical scattering by a sphere in Chapter 9, where we derive the famous integral representation of the solution to Maxwell's equations called the Stratton–Chu formula. In addition, we derive classical series representations of the solution of Maxwell's equations. These are used in Chapter 10 to derive a semi-discrete method for the scattering problem utilizing the electromagnetic equivalent of the Dirichlet to Neumann map. A fully discrete domain-decomposed version of this algorithm is proposed and analyzed in Chapter 11. The methods in Chapters 10 and 11 have the disadvantage of needing a truncated domain with a spherical truncation boundary. Obviously, using this method, high aspect ratio scatterers would require a domain with a large volume and, hence, large computational cost. Therefore, in Chapter 12 we turn to a coupled integral equation and finite element method due to Hazard and Lenoir [159] and Cutzach and Hazard [111]. In this method the Stratton–Chu formula is used to represent the solution outside the scatterer and simultaneously the finite element method is also used on a truncated domain extending outside the scatterer. There is thus a region where both methods represent the solution. It has to be admitted that this overlapping scheme is not the standard one in widespread use. I prefer this method because it avoids computing singular integrals and provides the basis for an alternating Schwarz iterative scheme for solving the problem. Readers interested in the more standard approach should consult the book of Jin [177] and the paper of Hiptmair [163].

There are of course many more problems associated with the finite element discretization of Maxwell's equations than those discussed in Chapters 7–12. In particular, the matrix problem resulting from the discretization of the Maxwell system is indefinite (regardless of the frequency of the radiation). Thus, the solution of this linear system (which is large and sparse) presents a serious challenge. Indeed, an efficient solution of this linear system is perhaps the main challenge currently facing finite element analysis of scattering problems. We discuss this problem in Chapter 13. This chapter also contains shorter discussions of a number of other practical aspects of the solution of Maxwell's equations. For example, we discuss the sensitivity of the error in the calculation to the frequency of the radiation and explain the need for a "sufficiently fine" grid compared to the wavelength of the radiation. We also consider *a posteriori* error estimation and the extraction of the far field pattern of the scattered wave from a knowledge of the near field. In addition, we examine the domain truncation problem further and, in particular, touch on the perfectly matched layer and infinite elements. These topics are much less well understood from the theoretical point of view than the error analysis presented earlier in the book.

The final chapter (Chapter 14) of the book hardly fits with the title, but since inverse problems are my main reason for studying scattering theory I cannot resist a brief introduction to inverse scattering. Besides its intrinsic interest, the chapter provides an example of the application of some of the analytical results derived earlier in the book.

There are a number of books that overlap to a greater or lesser extent with this work. The electrical engineering books of Jin [177] and Silvester and Ferrari [272] provide much more detail on coding finite element methods and, of course, more details of engineering applications. Thus, they complement my book rather well, with the book of Jin being most relevant because it focuses on edge elements. From the point of view of scattering theory in a variational setting, the book of Cessenat [73] is very useful but does not deal with numerical methods or (in the main) Lipschitz domains. Similarly, the book of Colton and Kress [94], although a vital source for much of the basic material in this book, uses a function space setting different from the one used here. In addition, finite element methods are not tackled. Perhaps closest to this book is the book of the founding father of this area, Professor Nédélec [236]. However, the emphasis of Nédélec's book is different in that he does not focus on finite element methods. Finally, although not a book, the massive survey article of Hiptmair [164] deserves mention. This article covers much of the material in Chapters 4 – 7 but at a more sophisticated level using discrete differential forms. In the same way as the book of Jin complements my book from the point of view of implementation, so does Hiptmair's article complement my presentation of finite elements and cavity problems.

Some comment needs to be made about the bibliography and references. I have roughly 300 references and have tried very hard to reference basic papers in the field. One area where the references are somewhat scarce is to the practical engineering literature. This does not represent a lack of enthusiasm for that literature. In fact, the widespread and successful engineering use of finite element methods and the need to buttress this success with a theoretical understanding are the motivations for this book. Since most of the theoretical work on finite elements has taken place in the mathematics literature, such papers appear in a disproportionate way in the bibliography.

Inevitably, there is an enormous amount of interesting material left out of this book. In essence, the contents are a reflection of my own research interests. In my defense, I can only quote Wittgenstein: "Whereof one cannot speak, thereof one must be silent" [297].

Of course I have tried to rid the book of as many typos as possible. But I am mindful that some bugs will have escaped detection. I plan to post any typos reported to me on the web page

`http://www.math.udel.edu/~monk/FEBook/index.html`.

In addition I will record there any interesting suggestions regarding arguments in the book (but I reserve the right to define what is "interesting"!).

Thanks are due to many people. My parents and the Falkland Island government gave me an excellent school education. My PhD adviser Rick Falk introduced me to finite elements, gave me tremendous encouragement as a graduate student, and even suggested the University of Delaware for postgraduate employment. In my professional life I have benefited tremendously from my collaboration and friendship with David Colton, who encouraged me to write this book. Outside the department, my family, and particularly my wife Ellen, have supported me and provided a wonderful antidote to depression and self-absorption. Particular thanks are also due to Pam Irwin, who cheerfully typed much of the book from my execrable notes, and to David Colton and Fioralba Cakoni who helped with the manuscript. Last, but by no means least, I would like to thank Dr Arje Nachman and the Air Force Office of Scientific Research for grant support which has made my research possible.

Newark P.M.
August 2002

CONTENTS

1

MATHEMATICAL MODELS OF ELECTROMAGNETISM

1.1 Introduction

In 1873 Maxwell founded the modern theory of electromagnetism with the publication of his *Treatise on Electricity and Magnetism*, in which he formulated the equations that now bear his name. These equations consist of two pairs of coupled partial differential equations relating six fields, two of which model sources of electromagnetism. It turns out that these equations are not sufficient to uniquely determine the electromagnetic field and that additional constitutive equations are needed to model the way in which the fields interact with matter. There is considerable flexibility in the constitutive equations. Because of this, we need to carefully state the problems to be analyzed in this book, and we start this chapter by summarizing the classical Maxwell equations governing an electromagnetic field in a linear medium. We then reduce this system to its time-harmonic form by assuming propagation at a single frequency. The time-harmonic Maxwell system will be the focus of this book. Besides Maxwell's equations, it is also necessary to describe appropriate physical boundary conditions. These include radiation conditions that select the outgoing field relevant to scattering problems.

Once the basic boundary value problem is formulated, it is often expedient to reduce the full Maxwell system to a simpler system relevant to the physical problem at hand. For example, it is often reasonable to assume that the electromagnetic field is time invariant or static. This reduces Maxwell's equations to a potential problem. Simpler models can also be derived at long and short wavelengths. We do not consider any of these reduced models here. We shall be concerned with approximating the time-harmonic Maxwell system for linear media in the "resonance region". By this we mean that the wavelength of the radiation is commensurate with the dimensions of features of the scatterer.

We end this chapter with a summary of the relevant boundary value problems from the point of view of this book. Our presentation, at this stage, is purely formal (we simply assume the existence of appropriate solutions) and follows the format of standard texts on electromagnetism, such as [274]. Later chapters will give a careful variational formulation of the equations in this chapter, followed by finite element methods.

First a word about notation: vectors are distinguished from scalars by the use of bold typeface (but this convention does not, in general, carry over to operators). Unless otherwise stated, vectors will all be three dimensional and either real (in $\mathbb{R}^3$) or complex (in $\mathbb{C}^3$). For example, $\boldsymbol{x} \in \mathbb{R}^3$ denotes position in three-space and has components x_1, x_2 and x_3 ($\boldsymbol{x} = (x_1, x_2, x_3)^\top$ where $\top$ denotes transpose). For two vectors $\boldsymbol{a} \in \mathbb{C}^N$ and $\boldsymbol{b} \in \mathbb{C}^N$ we define the dot product on $\mathbb{C}^N$ by

$$\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{j=1}^{N} a_j b_j.$$

The reason for not including complex conjugation in the dot product is that we will need to write down expressions like $\boldsymbol{\nu} \cdot \boldsymbol{E}$, where $\boldsymbol{\nu}$ is a real vector and $\boldsymbol{E}$ is complex. In this case we do not want to conjugate $\boldsymbol{E}$. Later, when we start to write down variational formulations, it will be important to recall that the dot product does not have complex conjugation built in. If $\boldsymbol{a} \in \mathbb{C}^N$ we define the Euclidean norm of $\boldsymbol{a}$ by $|\boldsymbol{a}| = \sqrt{\boldsymbol{a} \cdot \overline{\boldsymbol{a}}}$, where $\overline{\boldsymbol{a}} = (\overline{a}_1, \ldots, \overline{a}_N)^\top$ and $\overline{a}_j$ is the complex conjugate of a_j.

As usual in mathematics texts, $\mathrm{i} = \sqrt{-1}$, and j is just an integer variable. In our error estimates we shall use a generic constant C everywhere different. Apart from this, I have tried to avoid using the same symbol for two quantities (at least on the same page!).

1.2 Maxwell's equations

The classical macroscopic electromagnetic field is described by four vector functions of position $\boldsymbol{x} \in \mathbb{R}^3$ and time $t \in \mathbb{R}$ denoted by $\boldsymbol{\mathcal{E}}$, $\boldsymbol{\mathcal{D}}$, $\boldsymbol{\mathcal{H}}$ and $\boldsymbol{\mathcal{B}}$. The fundamental field vectors $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{H}}$ are called the electric and magnetic field intensities, respectively (we shall refer to them as the electric field and the magnetic field, respectively). The vector functions $\boldsymbol{\mathcal{D}}$ and $\boldsymbol{\mathcal{B}}$, which will later be eliminated from the description of the electromagnetic field via suitable constitutive relations, are called the electric displacement and magnetic induction, respectively.

An electromagnetic field is created by a distribution of sources consisting of static electric charges and the directed flow of electric charge, which is called current. The distribution of charges is given by a scalar charge density function ρ, while currents are described by the vector current density function $\boldsymbol{\mathcal{J}}$. Maxwell's equations then state that the field variables and sources are related by the following equations which apply throughout the region of space in $\mathbb{R}^3$ occupied by the electromagnetic field:

$$\frac{\partial \boldsymbol{\mathcal{B}}}{\partial t} + \nabla \times \boldsymbol{\mathcal{E}} = 0, \tag{1.1a}$$

$$\nabla \cdot \boldsymbol{\mathcal{D}} = \rho, \tag{1.1b}$$

$$\frac{\partial \boldsymbol{\mathcal{D}}}{\partial t} - \nabla \times \boldsymbol{\mathcal{H}} = -\boldsymbol{\mathcal{J}}, \tag{1.1c}$$

$$\nabla \cdot \boldsymbol{\mathcal{B}} = 0. \tag{1.1d}$$

Equation (1.1a) is called Faraday's law and gives the effect of a changing magnetic field on the electric field. The divergence condition (1.1b) is Gauss's law and gives the effect of the charge density on the electric displacement. The next equation, (1.1c), is Ampère's circuital law as modified by Maxwell. Finally, eqn (1.1d) expresses the fact that the magnetic induction $\boldsymbol{\mathcal{B}}$ is solenoidal. A table of SI units relevant to electromagnetism is given in Table 1.1.

The divergence conditions (1.1b) and (1.1d) are consequences of the fundamental field equations, (1.1a) and (1.1c), provided charge is conserved. Formally, this is shown by taking the divergence of (1.1a) and (1.1c) and recalling that $\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$ for any vector function $\boldsymbol{A}$. Hence

$$\nabla \cdot \frac{\partial \boldsymbol{\mathcal{B}}}{\partial t} = 0 \quad \text{and} \quad \nabla \cdot \frac{\partial \boldsymbol{\mathcal{D}}}{\partial t} = -\nabla \cdot \boldsymbol{\mathcal{J}}.$$

But if charge is conserved, ρ and $\boldsymbol{\mathcal{J}}$ are connected by the relation

$$\nabla \cdot \boldsymbol{\mathcal{J}} + \frac{\partial \rho}{\partial t} = 0, \tag{1.2}$$

and hence

$$\frac{\partial}{\partial t}\nabla \cdot \boldsymbol{\mathcal{B}} = \frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{\mathcal{D}} - \rho) = 0.$$

Thus if (1.1b) and (1.1d) hold at one time, they hold for all time. However, the fact that (1.1b) and (1.1d) are consequences of (1.1a) and (1.1c) for the continuous electromagnetic field does not mean that these divergence conditions can be entirely ignored when designing a numerical scheme to discretize (1.1). A successful scheme must produce a numerical approximation that in some sense satisfies discrete analogs of (1.1b) and (1.1d).

Either by using the Fourier transform in time, or because we wish to analyze electromagnetic propagation at a single frequency (e.g. if the source currents and charges vary sinusoidally in time), the time-dependent problem (1.1) can be reduced to the time-harmonic Maxwell system. If the radiation has a temporal frequency $\omega > 0$, then the electromagnetic field is said to be time-harmonic, provided

$$\boldsymbol{\mathcal{E}}(\boldsymbol{x}, t) = \Re\left(\exp(-\mathrm{i}\omega t)\hat{\boldsymbol{E}}(\boldsymbol{x})\right), \tag{1.3a}$$

$$\boldsymbol{\mathcal{D}}(\boldsymbol{x}, t) = \Re\left(\exp(-\mathrm{i}\omega t)\hat{\boldsymbol{D}}(\boldsymbol{x})\right), \tag{1.3b}$$

$$\boldsymbol{\mathcal{H}}(\boldsymbol{x}, t) = \Re\left(\exp(-\mathrm{i}\omega t)\hat{\boldsymbol{H}}(\boldsymbol{x})\right), \tag{1.3c}$$

Quantity	Units	Quantity	Units
Electric field intensity $\boldsymbol{\mathcal{E}}$	Vm^{-1}	Magnetic field intensity $\boldsymbol{\mathcal{H}}$	Am^{-1}
Electric displacement $\boldsymbol{\mathcal{D}}$	Cm^{-2}	Magnetic induction $\boldsymbol{\mathcal{B}}$	T
Electric current density $\boldsymbol{\mathcal{J}}$	Am^{-2}	Electric charge density ρ	Cm^{-3}

Table 1.1 *A table giving the SI units appropriate for electromagnetic quantities.*

$$\boldsymbol{\mathcal{B}}(\boldsymbol{x},t) = \Re\left(\exp(-\mathrm{i}\omega t)\hat{\boldsymbol{B}}(\boldsymbol{x})\right), \tag{1.3d}$$

where $\mathrm{i} = \sqrt{-1}$ and $\Re(.)$ denotes the real part of the expression in parentheses. Note that $\hat{\boldsymbol{E}}$ (and similarly other hat variables) are now complex-valued vector functions of position but not time. Some authors instead choose a time dependence of $\exp(\mathrm{i}\omega t)$. Of course, the choice is arbitrary and, provided it is used consistently, produces no difficulties. Our choice is fairly standard in the mathematics literature.

For consistency we also need the current density and charge density to be time-harmonic, so we assume

$$\boldsymbol{\mathcal{J}}(\boldsymbol{x},t) = \Re\left(\exp(-\mathrm{i}\omega t)\hat{\boldsymbol{J}}(\boldsymbol{x})\right),$$
$$\rho(\boldsymbol{x},t) = \Re\left(\exp(-\mathrm{i}\omega t)\hat{\rho}(\boldsymbol{x})\right).$$

Substituting these relations into (1.1) leads to the time-harmonic Maxwell equations:

$$-\mathrm{i}\omega\hat{\boldsymbol{B}} + \nabla\times\hat{\boldsymbol{E}} = 0, \tag{1.4a}$$
$$\nabla\cdot\hat{\boldsymbol{D}} = \hat{\rho}, \tag{1.4b}$$
$$-\mathrm{i}\omega\hat{\boldsymbol{D}} - \nabla\times\hat{\boldsymbol{H}} = -\hat{\boldsymbol{J}}, \tag{1.4c}$$
$$\nabla\cdot\hat{\boldsymbol{B}} = 0, \tag{1.4d}$$

where the time-harmonic charge density $\hat{\rho}$ is given via charge conservation (1.2) or by taking the divergence of (1.4c) and using (1.4b) as $\mathrm{i}\omega\hat{\rho} = \nabla\cdot\hat{\boldsymbol{J}}$ and hence can be eliminated from the equations.

Equations (1.4) give the time-harmonic Maxwell equations in differential form. Frequently, particularly in the physics literature, they are stated in integral form. As an example, consider (1.4a) and let S be a smooth surface in $\mathbb{R}^3$ with boundary ∂S and unit normal $\boldsymbol{\nu}$. Then, using Stokes theorem, we find that

$$\mathrm{i}\omega\int_S \hat{\boldsymbol{B}}\cdot\boldsymbol{\nu}\,\mathrm{d}A = \int_S(\nabla\times\hat{\boldsymbol{E}})\cdot\boldsymbol{\nu}\,\mathrm{d}A = \int_{\partial S}\hat{\boldsymbol{E}}\cdot\boldsymbol{\tau}\,\mathrm{d}s, \tag{1.5}$$

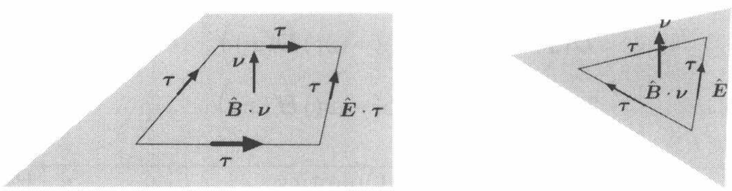

FIG. 1.1. For a surface S with normal $\boldsymbol{\nu}$ the integrated flux of $\hat{\boldsymbol{B}}$ normal to S is given by the integral of the tangential component of $\hat{\boldsymbol{E}}$ around the edges shown. Here we show schematics for a triangle and rectangle, two important surfaces from the point of view of numerical methods.

where $\boldsymbol{\tau}$ is the unit tangent to ∂S oriented by the right-hand rule relative to $\boldsymbol{\nu}$. In the integral formulation we see that $\hat{\boldsymbol{E}}$ is naturally associated to line integrals, whereas $\hat{\boldsymbol{B}}$ is naturally associated to surface integrals. For example, in Fig. 1.1 we show this when S is a triangle or a rectangle, two important cases that will appear later in the book.

Motivated by this integral formulation, finite difference schemes (in particular the famous FDTD scheme of Yee [301, 225]) usually associate the electric field $\hat{\boldsymbol{E}}$ with edges in a rectilinear mesh and the magnetic induction $\hat{\boldsymbol{B}}$ with faces. This is also the arrangement of discrete unknowns in a generalization of the rectangular finite difference scheme to tetrahedral grids called the co-volume scheme [214, 240, 241]. As we shall see in Chapter 5, we can also design finite elements that have a similar arrangement of unknowns. Finally, we note that (1.5) is also a starting point for the description of Maxwell's equations in terms of differential forms [164].

1.2.1 *Constitutive equations for linear media*

Equations (1.4) must be augmented by two constitutive laws that relate $\hat{\boldsymbol{E}}$ and $\hat{\boldsymbol{H}}$ to $\hat{\boldsymbol{D}}$ and $\hat{\boldsymbol{B}}$, respectively. These laws depend on the properties of the matter in the domain occupied by the electromagnetic field. We can distinguish three cases:

(1) *Vacuum or free space* In free space the fields are related by the equations

$$\hat{\boldsymbol{D}} = \epsilon_0 \hat{\boldsymbol{E}} \quad \text{and} \quad \hat{\boldsymbol{B}} = \mu_0 \hat{\boldsymbol{H}}, \tag{1.6}$$

where the constants ϵ_0 and μ_0 are called, respectively, the *electric permittivity* and *magnetic permeability*. The values of ϵ_0 and μ_0 depend on the system of units used. In the standard SI or MKS units

$$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{Hm}^{-1},$$
$$\epsilon_0 \approx 8.854 \times 10^{-12}\ \mathrm{Fm}^{-1}.$$

Furthermore the speed of light in a vacuum, denoted by c, is given by $c = \sqrt{\epsilon_0\mu_0}^{\,-1}$ ($c \approx 2.998 \times 10^8\ \mathrm{ms}^{-1}$) [274].

(2) *Inhomogeneous, isotropic materials* The most commonly occurring case in practice is that various different materials (e.g. copper, air, etc.) occupy the domain of the electromagnetic field. The medium is then called inhomogeneous. If the material properties do not depend on the direction of the field and the material is linear, we have

$$\hat{\boldsymbol{D}} = \epsilon \hat{\boldsymbol{E}} \quad \text{and} \quad \hat{\boldsymbol{B}} = \mu \hat{\boldsymbol{H}}, \tag{1.7}$$

where ϵ and μ are positive, bounded, scalar functions of position (we shall give a more careful description of these functions in Section 4.2).

(3) *Inhomogeneous, anisotropic materials* In some materials the electric or magnetic properties of the constituent materials depends on the direction of the