Mathematical Methods in the Physical Sciences

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Mathematical Methods in the Physical Sciences

To Ralph

Preface

This book is particularly intended for the student with one year of calculus who wants in a short time to develop a basic competence in each of the many areas of mathematics he will need to use in courses in physics, chemistry, and engineering. It may also be used effectively by a more advanced student to review half-forgotten topics and learn new ones, either in independent study or in a class. Although the book was written especially for students of the physical sciences, a student in any field (say, mathematics or mathematics for teaching) may find it useful to survey many topics or to obtain some knowledge of areas he does not have time to study in depth. Since theorems are stated carefully, such a student should not have to unlearn anything in his later work.

The question of proper mathematical training for students in the physical sciences is of concern to both mathematicians and those who use mathematics in applications. Mathematicians are apt to claim that if a student is going to study mathematics at all, he should study it in careful and thorough detail. For the undergraduate physics, chemistry, or engineering student, this means either learning more mathematics than a mathematics major, or learning a few areas of mathematics thoroughly and the others only from snatches in science courses. The second alternative is often advocated; let me say why I think it is unsatisfactory. It is certainly true that motivation is increased by the immediate application of a mathematical technique; however, there are a number of disadvantages. The discussion of the mathematics is apt to be sketchy since that is not the primary concern. The student is faced simultaneously with learning a new mathematical method and applying it to an area of science that is also new to him. Frequently the difficulty in comprehending the new scientific area lies more in the distraction caused by poorly understood mathematics than it does in the new scientific ideas. Finally, the student may meet what is actually the same mathematical principle in two different science courses without recognizing the connection, or even learn apparently contradictory theorems in the two courses! For example, in thermodynamics he learns that the integral of an exact differential around a closed path is always zero. In electricity or hydrodynamics, he runs into $\int_0^{2\pi} d\theta$, which is certainly the integral of an exact differential around a closed path but is not zero! Now it would be fine if every science student could take the separate mathematics courses in differential equations (ordinary and partial), advanced calculus, linear algebra, vector and tensor analysis, complex variable, Fourier series, probability, calculus of variations, special functions, etc. However, most science students have neither the time nor the inclination to study that much mathematics, yet they are constantly hampered in their science courses for lack of the basic techniques of these subjects. It is the intent of this book to give such students enough background in each of the needed areas so that they can cope successfully with junior, senior, and beginning graduate courses in the physical sciences. It is to be hoped, also, that some students will be sufficiently intrigued by one or more of the fields of mathematics to pursue it further.

It is clear that something must be omitted if so many topics are to be compressed into one course. I believe that two things can be left out without serious harm at this stage of a student's work, namely, generality, and detailed proofs. Stating and proving a theorem in its most general form is important to the mathematician and to the advanced student, but it is often unnecessary and may be confusing to the more elementary student. This is not in the least to say that the science student has no use for careful mathematics. The scientist, even more than the mathematician, needs careful statements of the limits of applicability of his mathematical processes, because he cannot so easily supply his own proofs of the accuracy of his work. Consequently I have endeavored to give accurate statements of the needed theorems, although often for special cases or without proof. Interested students can easily find more detail in textbooks in the special fields.

The existing texts on this subject almost invariably assume a degree of mathematical sophistication not yet reached by the student who has just completed one year of calculus. Yet such a student, if given simple and clear explanations, can master these techniques in rapid succession. (He not only *can*, but will *have to* one way or another if he is going to pass his junior and senior physics courses!) Such a student is not ready for detailed applications—these he will get in his science courses—but he does need and want to be given some idea of the use of the methods and some simple applications. This I have tried to do for each new topic. Many existing texts are slanted toward some particular field and contain a good many chapters on advanced topics in that special field. It is the purpose of this text to cover in simple fashion just the basic methods.

In developing this material for my own classes over the last five years, I have found that students with a background of one (or one and a half) years of calculus (or calculus and analytic geometry) can cover twelve or thirteen of the fifteen chapters in a one-year course meeting three times a week. With such students, I usually cover the first six chapters in the first semester, and a selection of the rest of the book the second semester. If differential equations is covered in a separate course, Chapter 7 may serve as a reference chapter. Chapter 9 may also be used satisfactorily as a reference chapter as needed (for gamma functions in connection with Bessel functions in Chapter 12; for the error function in Chapter 15, etc.) With students having more background (either advanced undergraduates or graduate students needing a brush-up course in the more advanced topics), I find it very satisfactory to start with Chapter 8 (or any desired later chapter) and refer back to earlier chapters as needed.

Although the chapters are not independent, a number of rearrangements of the material are possible. With very little difficulty, it is possible to start with Chapter 3 or Chapter 4 or Chapter 5 instead of Chapter 1. Chapter 4 and especially Chapter 5 are good starting places for a class which needs the motivation of many physical applications. Assuming just a little knowledge of partial differentiation and determinants, there is no trouble in starting with Chapter 5. If Chapter 4 precedes Chapter 1, the short section on two-variable power series can be omitted until later. The class should return to Chapters 1 and 2 before going on to Chapters 6 and 7. Chapter 15 is almost independent of the rest of the book and might reasonably be covered much earlier if desired. The following background is used in the other chapters:

For Chapter 8: Differential equations (Ch. 7).

For Chapter 9: Power series (Ch. 1); differentiating integrals (Ch. 4); Lagrange's equations (Ch. 8), used in one example.

For Chapter 10: Matrices (Ch. 3); vectors (Ch. 5); Lagrange's equations (Ch. 8).

For Chapter 11: Power series (Ch. 1); complex numbers (Ch. 2); partial differentiation (Ch. 4); vectors and vector theorems (Ch. 5); transformations (Ch. 10), used in one discussion.

For Chapter 12: Power series (Ch. 1); complex numbers (Ch. 2); partial differentiation (Ch. 4); differential equations (Ch. 7); Lagrange's equations (Ch. 8); gamma function (Ch. 9); background of Chapters 3, 5, 6, 10, used in discussion of eigenvalues and of orthogonal functions.

For Chapter 13: Fourier series (Ch. 6); complex variable (Ch. 11).

For Chapter 14: Legendre polynomials, Bessel functions, orthogonal functions (Ch. 12); Fourier series (Ch. 6); integral transforms (Ch. 13) used at the end of the chapter.

The only way to learn to apply mathematics to the solution of problems is to solve many problems. Consequently a large selection of problems—some applications and some just on the mathematical method itself—are included at the ends of the chapters.

Mary L. Boas

April, 1966

To the Student

As you start each topic in this book, you will no doubt wonder and ask "Just why should I study this subject and what use does it have in applications?" There is a story about a young mathematics instructor who asked an older professor "What do you say when students ask about the practical applications of some mathematical topic?" The experienced professor said "I tell them!" This text will try to follow his advice. However, you must on your part be reasonable in your request. It is not possible in one book or course to cover both the mathematical methods and very many detailed applications of them. You will have to be content with some information as to the areas of application of each topic and some of the simpler applications. In your later courses, you will then use these techniques in more advanced applications.

One point about your study of this material cannot be emphasized too much: To use mathematics effectively in applications, you need not just knowledge, but *skill*. Skill can be obtained only through practice. You can obtain a certain superficial *knowledge* of mathematics by listening to lectures, but you cannot obtain *skill* this way. How many students have I heard say "It looks so easy when you do it," or "I understand it but I can't do the problems!" Such statements show lack of practice and consequent lack of skill. The only way to develop the skill necessary to use this material in your later courses is to practice by solving many problems. You will find both drill problems and harder more challenging ones at the ends of the chapters. You should not feel satisfied with your study of a chapter until you can solve a reasonable fraction of these problems.

M. L. B.

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