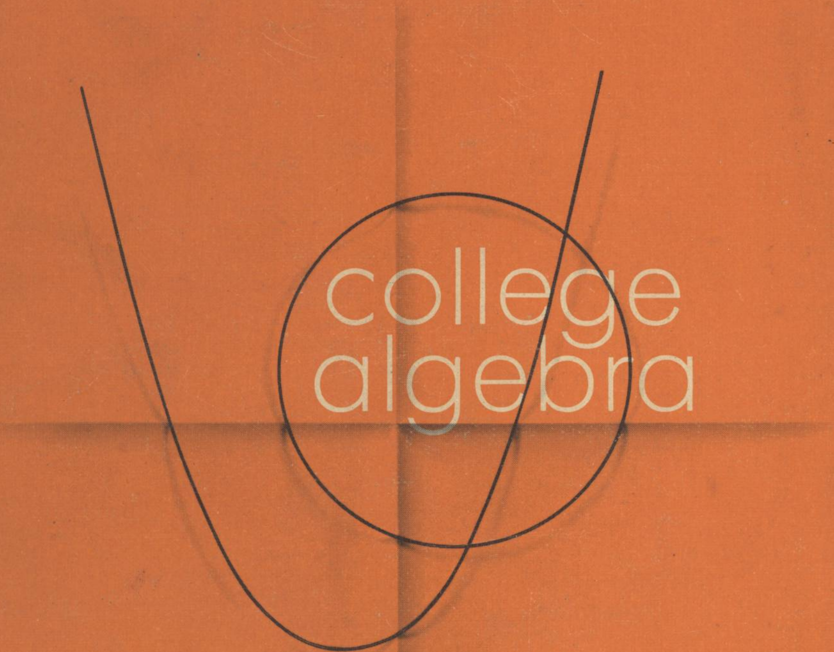
college algebra



college algebra

CARL H. DENBOW
Ohio University

HARPER & ROW, NEW YORK, EVANSTON, AND LONDON

to Stefania, Carl, Stef, Jane, and Signe

College Algebra

For information address Harper & Row, Publishers, Inc., 49 East 33rd Street, New York, N.Y. 10016.
LIBRARY OF CONGRESS CATALOG CARD NUMBER: 79-98199

Crawford

college algebra

preface

HOW TO USE THIS BOOK. Preparation for modern courses in calculus, linear algebra, and probability requires a thorough algebraic background and also presupposes a careful introduction to the rudiments of functions, including domain and range analysis, the study of composites, and the inverses of functions and relations. This book, planned for students who have completed two years of high school algebra and one year of high school geometry or the equivalent, covers these topics and others mentioned below.

Review is provided in extensive problem sets in the early chapters. New motivations are given by asking, for example, whether the solution sets involved have certain properties, and by using other devices of set language.

The more difficult problems occur, in general, at the end of each list. Proofs are sometimes omitted in the text, with an indication that a proof is requested in the problem section. The inclusion or omission of these harder problems and proofs will depend, of course, on the level of rigor desired.

Answers to nearly all odd-numbered problems are found in the back of the book. An *Instructor's Manual* supplement with answers and partial solutions to even-numbered problems and additional teaching suggestions is available. A *Student Manual* with self-help review material and tests is also available separately.

Chapter 1 is pedagogically important but introductory and can be covered quickly by classes that are well prepared. Chapter 2 presents a carefully motivated, rigorous derivation of the properties of the rational number system, as the "smallest" infinite field. In order to meet a wide diversity of backgrounds, mature approaches are used in Chapter 2 that should challenge all but the ablest (all but those ready for calculus). At the same time, the treatment is detailed enough to provide a good foundation for willing students of minimal preparation. There are many short proofs in this chapter, from which each instructor will make his own selection. Finite counterexamples are a special feature—for instance, an off-beat system in which division by zero is possible, and another in which adding like terms is invalid.

A short section of calculus at the beginning of Chapter 3 is presented intuitively in order to motivate the development of the real number system. Together with an unusual approach to the preponderance of irrationals over rationals, it prepares for the completeness postulate (section 3). Neither topic, however, is a prerequisite for the introduction of this postulate. The first part of secton 2 treats the topic of one-to-one correspondence, and should not be omitted.

A precalculus course should also stress Chapters 4 through 7 and, if time permits, Chapter 10, which defines limits of sequences. The treatment of composite and inverse functions given in Chapter 4 is unusually detailed, as is the treatment of systems with parameters in Chapter 5. These topics meet a need sometimes slighted in elementary courses: namely, they combine nontrivial manipulatory practice with increasing fluency in functional thinking.

The topics introduced in Chapters 8, 9, and 10 can be freely selected or omitted, except that addition properties of vectors are used in introducing complex numbers, and the inner product is needed in the treatment of matrices. Also, some problems in probability draw upon permutations. The section on limits of sequences in Chapter 10 is required for defining rigorously the sum of an infinite geometric progression.

NOTATION. *Mapping notation* such as

$$f : x \rightarrow y = x^3 - x$$

is used from Chapter 4 on, interchangeably with

$$f = \{(x,y) \mid y = x^3 - x\}$$

The student is likely to meet both in nonmathematical texts, without clear definition, and he will see other wordings, such as "$f(x)$ is a function." The variety of notations and abbreviations that he needs can be introduced, we believe, so that they reinforce each other; somewhat as, on another level, the use of a variety of symbols for variables has pedagogical value.

The concept "x approaches a" is used very sparingly in this book. Consequently, we reserve the arrow to mean *maps into* and use the symbol $\dashrightarrow$ for *approaches.*

The *study of functions,* begun in the early chapters, is carried further in Chapter 4, which includes graphing techniques and the algebra of functions, and is continued in the theory of systems of equations and of polynomial equations. Sequences are defined as functions on the natural numbers, and probability is introduced as a set function.

We use *norm function* and *absolute value function* as synonymous, in the real number system, preferring the former in the early stages. This usage emphasizes the technical nature of the concept, and experience indicates that it lessens errors of the type $|x| = x$.

Every function is a relation, of course, and every (binary) relation has an inverse, obtainable by interchanging the elements in each of its ordered pairs. It is rather common practice, however, with the convenient algebra of inverse functions in mind, to redefine the concept of inverse and to refer only to one-to-one functions as having inverses, identifying *inverse*, in this context, with inverses that are functions. We have been particularly careful to avoid confusing these meanings and have found it convenient to introduce the term *subfunction* for the restriction of a function to a domain on which it is one to one.

The use of set language offers the text writer his best vehicle for presenting many parts of mathematics meaningfully. The intrusion, into a manipulative chain of equations, of sets like

$$\left\{(x,y) \mid y = \frac{2}{x-3}\right\}$$

accompanied by questions about domain or range, shifts attention from an equation as an isolated object of manipulation to its solution set or the function that it defines; devices of this kind are used to help develop skill in mathematical analysis.

ACKNOWLEDGMENTS. I wish to express my sincere appreciation to many colleagues and students who have contributed suggestions and criticisms, including C. H. Anderson, R. K. Butner, Pablo Cue, K. E. Eldridge, V. Goedicke, S. J. Jasper, J. A. Karns, David Ladd, R. L. Lifsey, Jr., Armand Luhahi, Mrs. Charles Ramseth, H. Shankar, D. Wegener, Marty Weingold, and M. Wyzalek. The editorial staff of Harper and Row, and reviewers selected by them, were more than usually helpful as to content, style, and organization. I am particularly grateful to Frankie LaBorwit, James Martin, and Blake E. Vance, mathematics editor. Finally, my thanks to the administration of Ohio University for their helpfulness, and in particular for their establishment of a sabbatical leave system.

Three paragraphs of text and a dozen scattered problems are taken verbatim from my earlier book, *Foundations of Mathematics* (Harper &

Row, 1959), with the kind permission of my coauthor, Victor Goedicke. Part of its chapter on problem solving also appears here. That so little is reusable now is a reminder of the changes that have taken place in mathematics curricula; presentations needed a decade ago to bridge the gap between the so-called old and new mathematics either have been transferred to the high school curriculum or have been abandoned completely.

For errors found in this book, I am responsible, and I hope that readers will call them to my attention and continue giving helpful suggestions, as they did when *Foundations* appeared.

to the reader

One of the pleasures in store for the student of mathematics is the fascination of seeing simple little ideas—ideas that may have become stale with time or disuse—reshaped and guided into mathematical form, to emerge with new freshness and vitality. It is good to observe that an idea, sometimes an obscure one, can be transformed mathematically to play a major role in a new pattern of thought.

As you read this book several examples of this kind of transformation of a concept will be noted, including recurrence (page 48 ff.) and symmetry (page 57). We mention here an instance of a different kind, in which the idea we set out to capture mathematically is one that is already rich in significance. Consider the concept of pairing. When we observe that *something* is like *something else*, or when we judge *this* to be better than *that*, or decide that A is the result of B, we are giving three illustrations of the notion of pairing or associating. Clearly it is a rather general concept! How can mathematics, that great reshaper of ideas, handle an idea as broad as this? We can answer, with only a touch of exaggeration: That is what this book and many higher-level ones are all about. A more specific answer will begin to emerge on pages 25 and 34, and the reclothed simple notion of pairing will grow steadily in scope and majesty to the final pages of the last chapter.

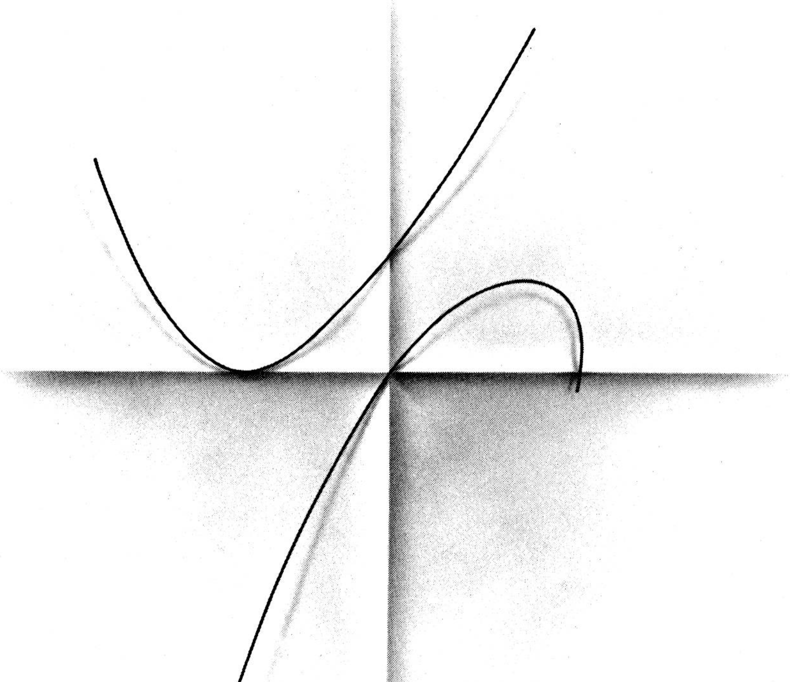
In using this book, the index pages should be consulted unsparingly. If you need to recall the meaning or application of a term, (solution set, composite function, ring, or other), locate its definition and find discussions of its use by checking the index. Follow the same procedure when you wish to review a process or topic (linear programming, vectors, etc.). For the meaning of a symbol, such as $\twoheadrightarrow$ or J^*, look in the symbol index on the inside covers of this book.

If you are using this book as a text in a course, we hope you will keep it after the course is ended and read it from time to time to refresh your ideas. Mathematics, more than some subjects, can grow stale from disuse. When you study calculus you may want to review, in Chapter 4 of this book, the concept of an inverse function, or in Chapter 2, the properties of inequalities; or other topics that you will find discussed in a very terse way, if at all, in your calculus text. If you note, in this book, a detailed discussion of some "fine point," why not check off the item in the book's index or add a notation of your own so that you can quickly locate and reread this passage when the need arises?

contents

part 1

fundamental concepts and techniques



chapter 1

number systems, sets, and functions

The purpose of this book is to provide the explanations, illustrative material, and exercises you need to acquire a good working proficiency in a new language, mathematics. We call it a new language even though it was flourishing at the time of Newton and did not sound strange to Archimedes. Only in recent years, however, has mathematics become the vehicle of discovery and communication in so many disciplines that it must rank as a working world language.

In the latter part of Chapter 1 and in Chapter 2 we will examine the mathematical method. It goes by various names. The method of models, of mathematical languages, the postulational method—each of these describes one of its major aspects. When you have learned it—as you will, if you do the problems provided—you will have the background for further study in mathematics and the admission ticket to economics, systems analysis, and cosmology, to engineering and genetics and computer science. You can extend the list much further.

Read this book with pencil and paper at hand; mull over the ideas and drill yourself in the language. We shall set a slow, deliberate pace, and, if you maintain your efforts, you will acquire a firm base in one of the most creative of man's intellectual achievements.

1. The Equals Relation

The concept of equals illustrates how a notion widely used in everyday language takes on new roles when woven into a mathematical system.

Consider the statement: "Dr. Roger Bannister is the first man known to have run a mile in less than 4 minutes." The word *is* here has the meaning of *equals*; it asserts that Dr. Roger Bannister and the first man known to have run a mile in less than 4 minutes are one and the same man. This is not the only meaning of *is*. Consider the statement: "Caesar is ambitious." Its meaning can be expressed in the language of sets: "Caesar belongs to the set of ambitious persons."

It is typical of the mathematical method that it separates these meanings and uses different symbols for them. We use the sign $=$ for the *equals* meaning of *is*, as in the Bannister example, and we write $x \in A$ to indicate that x belongs to a set A (is a member of A) as in the Caesar example.

Having symbolized $=$, we then proceed to generalize its meaning in new directions. We use it to make *affirmative statements*, as in "$1 + 3 + 5 + 7 = 4^2$," or "for every real number x, we have $x(x - x) = 0$." We use it also when we are *investigating possibilities*, as in the two examples below. When we write

$$(x + 2)^3 = x^3 + 8 \qquad \text{(Equation a)}$$

we are asking for what xs, if any, the two members of the equation are equal. Answer: $x = 0$, $x = -2$, as can easily be determined by methods discussed later. The statement

$$(x + 2)^3 = (x + 1)^3 \qquad \text{(Equation b)}$$

turns out not to be true for any real number x. It is an *equation written tentatively* that upon investigation proves to be always false. The corrected version of (b) is

$$(x + 2)^3 \neq (x + 1)^3$$

for all real numbers x, where $\neq$ means "is not equal to." We will have occasion to write and investigate several *no-solution* equations as we proceed.

PROPERTIES OF EQUALS. It is typical of mathematics that it should search out and enumerate the basic properties of the equals relation. Here are four of them:

E 1. For any object a, $a = a$ (the reflexive property).

E 2. If $a = b$, then $b = a$ (property of symmetry).

E 3. If $a = b$ and $b = c$, then $a = c$ (property of transitivity).

E 4. Any quantity may be substituted for its equal.

E 4 is a very strong statement from which E 1, E 2, and E 3 can be derived. For future reference, however, we prefer to list all four properties.

A REMARK ON NOTATION. Instead of writing a chain of equations such as

$$a = b,\ b = c, \text{ and } c = d$$

we will often abbreviate and write

$$a = b = c = d$$

Problems

State which property of equals is needed to get

1 $u = w$ from $u = v = w$

2 $2a + 5 = b$ from $2a + 5 = c$ and $c = b$

3 Write six equations that can be derived from the set

$$x = y,\ y = z,\ z = v, \text{ and } v = w$$

by using the transitive property E 3.

4 If we agree that every man is his own brother, then the relation of brotherhood is reflexive, symmetric, and transitive, just as equals is. (It is transitive because, if A is the brother of B and B is the brother of C, then A is the brother of C.) Which of the following relations are reflexive, symmetric, and transitive: Classmate of, perpendicular to, taller than, of the same generation as, father of? Reflexivity—of brotherhood for example—is defined more precisely in section 5.

2. Sets[1]

A *set* is any well-defined collection[2] of objects, which are called its *elements* or *members.* The elements may be physical (a set of chairs, people, or planets) or conceptual (the set of rational numbers or the set of integers greater than 6 and less than 90) or a mixture. The set of all counting numbers $\{1,2,3,\cdots\}$ is an example of what is called an *infinite* set, while a set of n elements, for any specific counting number n, is called a *finite* set. The dis-

[1] Well-prepared students may skim this section rapidly, working some of the problems.

[2] This is a discussion of *set,* not a definition.

tinction is a fundamental one, which we shall later define precisely. We consider a set with no elements and call it the empty set, or null set, and denote it by $\emptyset$, which is a letter from the Norwegian alphabet. You need not pronounce it; read it *the empty set.* An example of $\emptyset$ is the set of all rational numbers greater than 1 that are equal to their own squares. All empty sets are equal, by definition.

A standard notation for a set uses curly braces, $\{\ \}$, as above. We use capital letters also as names of sets; thus the sets $\{0,1,2,3,4,5\}$ and $\{2,4,6\}$ may be denoted by A and B.[3] The members of B are 2, 4, and 6; we also say, more briefly, that they are *in* B. The symbol for elementhood or membership is $\in$; thus we can write $2 \in B$, $4 \in B$, and $6 \in B$ (read "2 is a member of B" or "2 is in B"). We write $8 \notin B$ to indicate that 8 is not in B.

We list some sets of numbers that will be analyzed in detail later.

- We denote by J^* the set of *integers*

$$J^* = \{\cdots,-2,-1,0,1,2,3,\cdots\}$$

The three dots are a standard notation meaning *and so on endlessly.* An integer that is a multiple of 2 is called an *even* integer; it is expressible in the form $2n$, where n is an integer. Numbers of the form $2n + 1$, where n is an integer, are called *odd* numbers. An integer a is said to be *divisible* by an integer b, where $b \neq 0$, if there is an integer c such that $a = bc$; that is, if a is the product of b and c. Thus, 6 is divisible by 3, since $6 = 3 \cdot 2$.

- An important subset of J^* is the set of *natural numbers* or *positive integers,* also called the *counting numbers.* We call this set N^*.

$$N^* = \{1,2,3,4,\cdots\}$$

- The set of rational numbers, Q^*, is the set of all *ratios* or *quotients* a/b (examples: $-7/2$, $3/5$, $111/11$, etc.) where $a \in J^*$ and $b \in J^*$ and $b \neq 0$. Note that any integer such as -5 can be written also as a ratio; $-5 = -5/1 = 15/-3$, etc. Therefore Q^* is an extension of the system J^*.

- The *real numbers,* R^*, includes the set Q^*, together with irrational numbers (see Chapter 3) such as $\sqrt{2}$, $-\sqrt[3]{5}$, and $-7 + \pi$.

- The complex numbers, C^*, are numbers of the form $a + bi$, where a and b are in R^* and $i = \sqrt{-1}$. (The subset $\{a + bi \mid b \neq 0\}$ is

[3] Here we are giving illustrations of the notion of set, taking it as an undefined term.