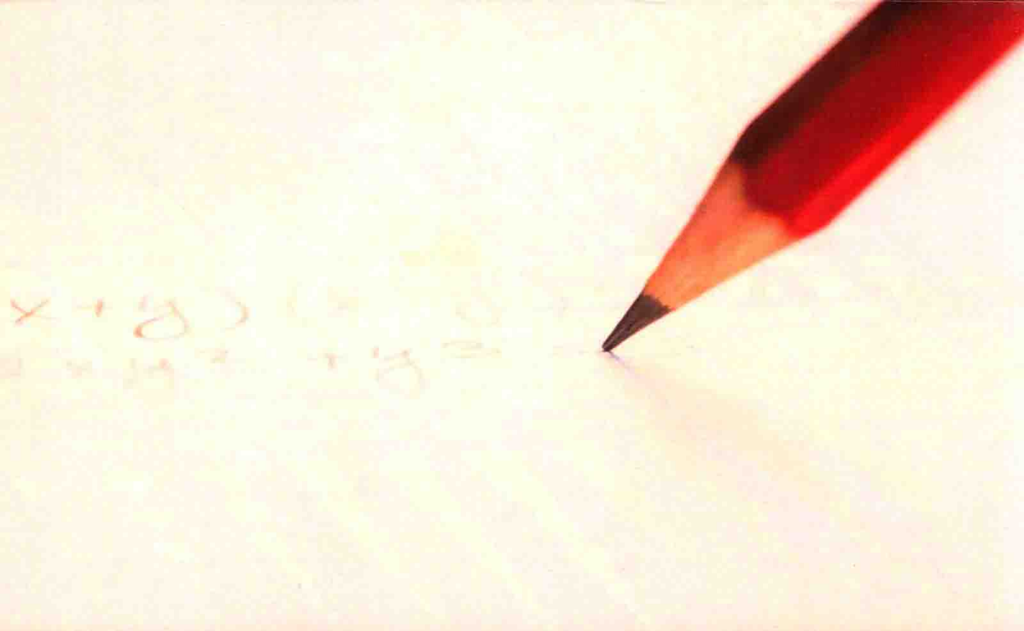

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Generalized fuzzy K-algebras

Theory, Methods and Computations

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Preface

Most of the traditional methods for formal modeling, reasoning and computing are crisp and precise. But real life data is not always crisp, and all descriptions can not be always expressed or measured precisely. In 1965, L. A. Zadeh proposed a new mathematical model known as Fuzzy Set Theory, which dealt with such type of real life data. The genuine necessity of such a new mathematical model stems from the fuzziness of the natural phenomena. Fuzzy sets have been applied in a wide variety of fields, including Computer Science, Medical and Life Sciences, Management Sciences, Social Sciences, Engineering and Theoretical Mathematics. A number of generalizations of Zadeh's fuzzy set theory are currently available in the literature. The interval-valued fuzzy sets, intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets, bipolar fuzzy sets, vague sets, type 2 fuzzy sets and grey sets are a few examples of these generalizations.

In 2003, we introduced a new kind of logical algebra $(G, \cdot, \odot, e)$, known as K-algebra, which is built on a group $(G, \cdot, e)$ by adjoining the induced binary operation $\odot$ on $(G, \cdot, e)$. In 2005, we, along with Y. B. Jun and E. H. Roh, first introduced fuzzy structures on K-algebras and investigated some of their properties. We then developed a fuzzy K-algebra with other researchers worldwide by applying generalizations of Zadeh's fuzzy set theory. We present in this book some of the results of our K-algebras and generalized fuzzy K-algebras, which have appeared in some of our published papers. We thank the researchers worldwide whose contributions are referenced in this book, especially L. A. Zadeh, K. T. Atanassov, John Mordeson, A. Rosenfeld, Y. B. Jun, W. A. Dudek, D. Coker, S. K. Bhakat, P. Das, W.-R. Zhang, R. Biswas, B. Davvaz, W. L. Gau and D. J. Buehrer. We are greatly thankful to K. P. Shum for his interest in our work, devoting his valuable time to read this book, and giving useful comments. We also thank the staff of VDM Publishing House Ltd, especially the Acquisition Editor, Tabassum Dowlut, for facilitating us throughout.

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Introduction

Algebraic structures play a prominent role in mathematics with a wide range of applications in many disciplines including theoretical physics, computer science, control engineering, information sciences, coding theory and topological spaces. Thus, algebraic structures provide sufficient motivation to researchers to review various concepts and results from the realm of abstract algebra in the broader framework of fuzzy setting.

The notion of a *K-algebra* $(G, \cdot, \odot, e)$ was first introduced by us in [60, 61]. A K-algebra is an algebra built on a group $(G, \cdot, e)$ by adjoining an induced binary operation $\odot$ on G which is attached to an abstract K-algebra $(G, \cdot, \odot, e)$. This system is in general non-commutative and non-associative adjoint with a right identity e, if $(G, \cdot, e)$ is non-commutative. For a given group G, the K-algebra is proper if G is not an elementary Abelian 2-group. Thus, whether a K-algebra is Abelian and non-Abelian purely depends on the base group G. We call *a K-algebra on a group G* as a $K(G)$-algebra due to its structural basis G. We along with Y. B. Jun and E. H. Roh first introduced the notion of a fuzzy K-algebra. We and other researchers worldwide developed fuzzy K-algebras. In this monograph, we have made an effort to present some results of the generalized fuzzy K-algebras. The brief description of the content of each chapter is given below.

In Chapter 1, we present a comprehensive review of fuzzy literature. In Chapter 2, we describe the theory of K-algebra, K-subalgebras and ideals of K-algebras. We discuss characterization theorems of K-algebras by using their left and right mappings. We also talk about the left K-algebra. In Chapter 3, we present K-homomorphisms of K-algebras and isomorphisms theorems. We present the subclasses of K-algebras in Chapter 4.

In Chapter 5, we discuss fuzzy K-algebras, fuzzy topological K-algebras and the concept of fuzzy ideals in detail. We also talk about the concept of T-fuzzy K-algebras. We discuss some new types of fuzzy K-algebras in Chapter 6. In Chapter

7, we describe the interval-valued fuzzy K-algebras including interval-valued fuzzy ideals of K-algebras. In Chapter 8, we explain in detail the concept of intuitionistic fuzzy ideals of K-algebras. We present the intuitionistic fuzzy topological K-algebras. We describe the concept of interval-valued intuitionistic fuzzy ideals of K-algebras in Chapter 9. In Chapter 10, we discuss a new generalized fuzzy subalgebra of a K-algebra called bipolar fuzzy K-subalgebra. We explain vague K-subalgebras in Chapter 11. Finally, we state some open-ended problems and applications of K-algebras in Chapter 12.

Contents

Chapter 1

Review of Fuzzy Literature

1.1 Historical background

In handling information regarding various aspects of uncertainty, non-classical logic (a great extension and development of classical logic) is considered to be a more powerful technique than the classical logic. The non-classical logic has nowadays become a useful tool in computer science. Moreover, non-classical logic deals with fuzzy information and uncertainty.

It is well known that the notion of a *fuzzy subset of a set* was first introduced by Zadeh [146] in 1965 as a method of representing uncertainty. Since then, fuzzy set theory has been developed in many directions by many scholars and has evoked great interest among mathematicians and computer scientists working in different fields of mathematics and computer science, including topological spaces, functional analysis, loops, groups, rings, semirings, hemirings, nearrings, vector spaces, differential equations, pattern recognition, robotics, computer networks, expert systems, decision making theory, and automation. In 1971, Rosenfeld [127] used the concept of a fuzzy subset of a set to introduce the notion of a fuzzy subgroup of a group. Rosenfeld's paper spearheaded the development of fuzzy abstract algebra.

After introducing the concept of fuzzy sets, we discuss a number of new theories treating imprecision and uncertainty. Some of these theories are extensions of the fuzzy set theory. In 1975, Zadeh [147] introduced the notion of *interval-valued fuzzy sets* as an extension of fuzzy sets in which the values of the membership degrees are intervals of numbers instead of the numbers. Interval-valued fuzzy sets provide a more adequate description of uncertainty than traditional fuzzy sets. It is therefore important to use interval-valued fuzzy sets in applications, such as fuzzy control. One of the computationally most intensive part of fuzzy control is defuzzification [117]. Since interval-valued fuzzy sets are widely studied and used, we describe briefly the work of Gorzalczany on approximate reasoning [90, 91], Roy and Biswas on medical diagnosis [128], Turksen on multivalued logic [135] and Mendel on intelligent control [117].

Atanassov [34] introduced the concept of *intuitionistic fuzzy sets* in 1983 as a generalization of fuzzy sets. Atanassov added a new component(which determines the degree of non-membership) in the definition of fuzzy set. The fuzzy sets give the degree of membership of an element in a given set (and the non-membership degree equals one minus the degree of membership), while intuitionistic fuzzy sets give both a degree of membership and a degree of non-membership which are more-or-less independent from each other, the only requirement is that the sum of these two degrees is not greater than 1. Intuitionistic fuzzy sets have been applied in a wide variety of fields including computer science, engineering, mathematics, medicine, chemistry and economics [37, 73]. With the above background, Atanassov and Gargov [34] introduced the notion of *interval-valued intuitionistic fuzzy sets* in 1989 which is a common generalization of intuitionistic fuzzy sets and interval-valued fuzzy sets.

The concept of *bipolar fuzzy sets* was first initiated by Zhang [155, 156] in 1994 as a generalization of fuzzy sets. Bipolar fuzzy sets are an extension of fuzzy sets whose membership degree range is $[-1, 1]$. In a bipolar fuzzy set, the membership degree 0 of an element means that the element is irrelevant to the corresponding property, the membership degree $(0, 1]$ of an element indicates that the element somewhat satisfies the property, and the membership degree $[-1, 0)$ of an element indicates that the element somewhat satisfies the implicit counter-property. Although bipolar fuzzy sets and intuitionistic fuzzy sets look similar to each other, they are essentially different from each other [111]. In many domains, it is important to be able to deal with bipolar information. It is noted that positive information represents what is granted to be possible, while negative information represents what is considered to be impossible. This domain has recently motivated new research in several directions. In particular, fuzzy and possibilistic formalisms for bipolar information have been proposed [78], because when we deal with spatial information in image processing or in spatial reasoning applications, this bipolarity also occurs. For instance, when we assess the position of an object in a space, we may have positive information expressed as a set of possible places, and negative information expressed as a set of impossible places. As another example, let us consider the spatial relations. Human beings consider "left" and "right" as opposite relations. But this does not mean that one of them is the negation of the other. The semantics of "opposite" captures a notion of symmetry rather than a strict complementation. In particular, there may be positions which are considered neither to the right nor to the left of some reference object, thus leaving some room for indetermination. This corresponds to the idea that the union of positive and negative information does not cover the whole space.

The notion of *vague set theory*, also generalization of Zadeh's fuzzy set theory, was introduced by Gau and Buehrer [89] in 1993. In most cases of judgment, evaluation is done by human beings (or by an intelligent agent) where there certainly is a limitation of knowledge or intellectual functionaries. Naturally, every decision-maker hesitates, more or less, on every evaluation activity. For example, to judge whether a patient has cancer or not, a doctor (the decision-maker) will hesitate to give his opinion because a fraction of his evaluation is in favor of truth, another fraction is in favor of falseness and the rest remains undecided to him. This is the fundamental philosophy behind the notion of vague set theory.

1.2 Review of fuzzy concepts

Definition 1.2.1. Let X be a non-empty set (to be called the universe of discourse or domain or universal set). A classical (crisp) set on X is a mapping whose codomain is 0 and 1., i.e., $f : X \to \{0, 1\}$.

Definition 1.2.2. Let X be a non-empty set. A fuzzy set on X is a mapping whose codomain is [0, 1], i.e., $\mu : X \to [0, 1]$. μ is called membership function, $\mu(x)$ is called degree of membership function of x.

OR

A fuzzy set on X can be defined as an object of the form:

$$\mu = \{(x, \mu(x)) : x \in X\},$$

where the function $\mu : X \to [0, 1]$ denote the *degree of membership*.

In the theory of fuzzy sets the membership degrees of elements range over the interval [0, 1]. The membership degree expresses the degree of belongingness of elements to a fuzzy set. The membership degree 1 denote that an element completely belongs to its corresponding fuzzy set, and the membership degree 0 denote that an element does not belong to the fuzzy set. The membership degrees in the interval $(0, 1)$ denote the partial membership to the fuzzy set. For applications of fuzzy sets, the readers are referred to [77, 154].

Definition 1.2.3. The fuzzy empty set and the fuzzy whole set in a set X are denoted by 0_X and 1_X, and defined as $0_X = \{x \in X : \mu(x) = 0\}$ and $1_X = \{x \in X : \mu(x) = 1\}$, respectively.

A fuzzy set operation is an operation on fuzzy sets. These operations are generalization of crisp set operations.

Definition 1.2.4. Let A and B be two fuzzy sets in the universe of discourse X, and μ_A and μ_B be the membership functions of A and B, respectively, then the operations over fuzzy sets are defined as follows:

- $A \subseteq B \Rightarrow \mu_A(x) \leq \mu_B(x)$,
- $A = B \Leftrightarrow \mu_A(x) = \mu_B(x)$,
- $A \cap B = \min\ (\mu_A(x), \mu_B(x))$,
- $A \cup B = \max\ (\mu_A(x), \mu_B(x))$,
- $A \cdot B = \mu_A(x) \cdot \mu_B(x)$,
- $A + B = \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x)$,
- $\overline{A} = 1\text{-}\ \mu_A(x)$

for all $x \in X$.

Remark. The following two laws do not hold in the theory of fuzzy sets:

(a) The law of excluded middle is not valid in fuzzy set theory, that is,

$$A \cup \overline{A} \neq 1_X$$

(b) Law of contradiction is not valid in fuzzy set theory, that is,

$$A \cap \overline{A} \neq 0_X$$

Definition 1.2.5. For $s \in [0, 1]$, the set $U(\mu; s) = \{x \in X : \mu(x) \geq s\}$ is called *upper level (s-cut)* of μ.

Definition 1.2.6. A fuzzy set μ in a group G is called a *fuzzy subgroup* of G if it satisfies:

(c) $(\forall x, y \in G)\ (\mu(xy) \geq \min\{\mu(x), \mu(y)\})$.

(d) $(\forall x \in G)\ (\mu(x^{-1}) \geq \mu(x))$.