PATH-INTEGRAL METHODS and their APPLICATIONS

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PREFACE

Quantum mechanics is a probabilistic theory while the classical mechanics is a deterministic one. The connection between the two was eluding for a long time till Feynman blended them through his famous path integral representation of the kernel of the Schrödinger equation. Since then the path integral has come a long way and continues to inspire both physicists and mathematicians. For physicists, it is a very powerful tool, both at the conceptual as well as computational level. For mathematicians, on the other hand, it is a priori a meaningless object and continues to provide a challenge for developing an appropriate framework within which it can be defined, understood and manipulated.

These aspects of Feynman path integrals have been a stimulant to us during our long association with this field. Also during this period of three decades the path integral literature has grown rapidly. The first classic book by Feynman is already more than thirty years old while another book by Schulman is almost a decade old. The third excellent book by Wiegel concentrates primarily on problems on polymer science. Thus a need to write yet another book on the subject was felt by us.

The book attempts to cover the recent developments in the field of path integrals with the emphasis on exact results and approximation schemes along with applications in various areas. A conscious effort has been put to make the book useful to specialists in the field and simultaneously transparent to non-specialists primarily interested in the practical applications of path integration.

The first chapter introduces the concept of Feynman path integral and its relation with allied areas like Brownian motion and Wiener integrals. The second chapter discusses the instantaneous quadratic Lagrangians and prepares the reader for the basic algebraic

manipulations inherent in a path integral treatment of a quantal problem. The third chapter deals with the path integration of the so called "two-time quadratic actions". The results derived in this chapter are particularly useful from the point of view of applications. The path integration in curved spaces forms the subject matter of chapter 4. In particular, the path integration in polar coordinates is treated in detail considering its importance in applications. The fifth chapter introduces the recently developed global and local time transformation techniques because of their potential in the context of path integration. The next chapter is concerned with the path integration methods to deal with the problems involving topological constraints. This chapter provides also an account of selected applications of these techniques.

Most of the literature on the subject have not considered the relation between invariants of motion and Feynman propagator and hence it was felt desirable to include a chapter on this theme. Chapter 7 discusses in detail how the knowledge of invariants of motion can be helpful in obtaining Feynman propagator.

The next three chapters are concerned with some approximation techniques frequently used in path integral applications. Notable amongst them is the so-called cumulant approximation method. A separate chapter is devoted to this method and its applications. The chapter on perturbation method has been written with a slightly different perspective. The major question which we pose here concerns the summability of the series rather than the conventional applications which can be found in almost all text books on the subject. The chapter on semiclassical analysis has been added primarily for the sake of completeness. An extensive and excellent discussion of this topic is already available in published form.

A significant departure from the earlier published books is the inclusion of a chapter on numerical methods of computing path integrals. In particular, Monte Carlo methods have been discussed in detail in chapter 11.

The last chapter presents briefly the difficulties encountered in providing a proper mathematical interpretation to Feynman path integral

and some of the attempts in resolving them. Most of the discussion has been deliberately kept at a qualitative level to make it more transparent. The more curious reader may find the references listed at the end of chapter 12 useful for more rigorous treatment.

Writing of this book has taken more than two years. Stimulating discussions with several colleagues have helped in bringing the book to its present form. It is a pleasure to thank them all. In particular we acknowledge constant encouragement from Dr. R. Chidambaram all throughout this trying period. Lastly, we must express our sincere thanks to Mrs. Smita Khandekar for her assistance in typing and editing of the present manuscript.

Authors

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CHAPTER 1

INTRODUCTION TO PATH INTEGRALS

1.1. Introduction

This book is on path integrals and their applications to physics at large. Naturally we begin with Feynman, who first introduced the concept to physicists in his new space-time formulation of non-relativistic quantum mechanics published in his classic 1948 paper in Reviews of Modern Physics. Feynman himself established that this third formulation of quantum mechanics, the path integral approach, is equivalent to the usual formulations of Schrödinger as well as that of Heisenberg and Dirac. Incidentally, three independent mathematical disciplines are associated with the three formulations of quantum mechanics. While Heisenberg-Dirac method relies on "algebra", Schrödinger's approach is based on differential equations and hence uses "analysis". Feynman's method on the other hand is based on "geometry". This geometrical way of expressing the quantum superposition principle is intuitively appealing since it allows us to directly visualize the constructive or destructive interference arising from many different paths. Feynman himself attributed this multiplicity of possible descriptions of quantum phenomena to *our having captured key elements in our description of atomic phenomena and is an expression and representation of the simplicity of nature.*

In fact the notion of a path integral usually called a functional integral had been familiar to mathematicians much before Feynman. It was Volterra who used this idea in his work on functional calculus. A functional is to be considered analogous to a function of infinitely many variables. Calculations involving a functional are carried out by assuming it to be a function of a finite number N of variables and

subsequently letting $N \longrightarrow \infty$. The procedure is similar to the "time slicing prescription" used by Feynman. We shall encounter this many times later in the book. Early papers of Daniell deals with some attempts of integrating a functional over a space of functions and are reviewed by Kac. Subsequently Wiener introduced a proper measure-theoretic definition of an integral of a functional over a space of functions. The analogy between Feynman path integral and Wiener integral has been extensively discussed in the literature.

The major motivation of this chapter is to introduce the notion of a path integral from a physicist's point of view. We shall outline the path integral formulation of non-relativistic quantum mechanics in a way that Feynman presented in his 1948 paper and subsequently in the book by Feynman and Hibbs. We shall then discuss the alternative ways of looking at a path integral using the ideas of random walk, Brownian motion and Wiener measure.

1.2. Feynman Path Integral

Before we present Feynman's intuitive arguments for introducing the path integral formulation we may outline the scenario at his time. Quantum mechanics was traditionally based on the Hamiltonian formulation of classical mechanics. The rules governing the transition from classical to quantum description are summarized in Table 1.

Table 1 :Classical vs. Quantum Description

	Classical Mechanics	*Quantum Mechanics*
1. Variables:	x, p (c-numbers), $\{x,p\} = 1$	$\hat{x}$, $\hat{p}$ (operators), $[\hat{x},\hat{p}] = 1$
2. Hamiltonian:	$H(x, p)$	$\hat{H}(\hat{x}, \hat{p})$
3. Dynamical Law:	$df(x,p)/dt = \{f,H\}$	a) Heisenberg Eq : $i\hbar\frac{d\hat{f}}{dt} = [\hat{f},\hat{H}]$
	H. J. equation ?	b) Schrödinger Eq : $i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi$
		$\hat{H} = H(x, -i\hbar\partial/\partial x)$
4: Lagrangian:	$L(x, \dot{x})$	?

The classical description of a system is through a set of conjugate variables x, p satisfying the Poisson Bracket (PB) relation $\{x,p\} = 1$.

The dynamics is described by Hamilton's equations written in the general PB notation in Table 1. In quantum description these conjugate variables are replaced by non-commuting operators $\hat{x}$, $\hat{p}$ and a statement governing the time evolution of these operators. The commutation relation is $[\hat{x},\hat{p}] = i\hbar\hat{I}$, $\hat{I}$ representing the identity operator. Further, the dynamical law is

$$i\hbar\, d\hat{x}/dt = [\hat{x}, \hat{H}] \tag{2.1}$$

in Heisenberg picture which is reminiscent of classical PB theory. Alternatively the dynamical law may take the form of the Schrödinger equation

$$i\hbar\, \partial\psi/\partial t = \hat{H}\psi \tag{2.2}$$

where the quantum Hamiltonian operator $\hat{H}$ is obtained from the classical Hamiltonian $H(x, p)$ by the simple replacement $p \longrightarrow -i\hbar\partial/\partial x$. In all this formulation there remained an important gap between quantum and classical mechanics. The Lagrangian formulation, which had preceded the Hamiltonian approach in classical mechanics had practically no role in the quantum formulation. There was, however, one remote connection. This involved the derivation of classical Hamilton-Jacobi (HJ) equation from the Schrödinger equation by means of the transformation

$$\psi \longrightarrow C \exp(iS/\hbar) \tag{2.3}$$

where S is the classical action and C is real. In fact the substitution of (2.3) in Schrödinger equation

$$i\hbar\, (\partial/\partial t)\psi = (-\hbar^2/2m)\, (\partial^2/\partial x^2)\psi + V\psi \tag{2.4}$$

for a particle of mass m moving in a one-dimensional potential $V(x)$, yields the HJ equation

$$(\partial S/\partial t) + (1/2m)\, (\partial S/\partial x)^2 + V = 0 \tag{2.5}$$

if terms $O(\hbar/S)$ are neglected. If, however, these terms are retained, we obtain the continuity equation

$$\partial\rho/\partial t + \partial(\rho v)/\partial x = 0 \tag{2.6}$$

where $\rho = |\psi|^2 = C^2$ and $v = (\partial S/\partial x)/m$. Note that (2.3) with S determined from (2.5) represents just the solution of (2.4) in WKB approximation

valid for $\hbar \to 0$. This correspondence between Schrödinger equation and HJ equation in the semiclassical limit ($\hbar \to 0$) answers the first question raised in Table 1.

Regarding the second question in Table 1, it was Dirac who had first emphasized the possible importance of Lagrangian in quantum mechanics in one of his early papers and later in his book. Feynman exploited Dirac's remarks to arrive at his so-called Lagrangian formulation of quantum mechanics.

1.2.1. *Feynman formulation*

In standard quantum mechanics one assigns a complex probability amplitude $\psi(x,t)$ (called the wave function) with the position x of a particle at time t. For simplicity, we shall use notation corresponding to a single dimension, the generalization to more than one being trivial. The probability density for the particle to be found at x at time t is then simply given by $|\psi(x,t)|^2$. Instead, Feynman associated a probability amplitude with the "entire motion of a particle as a function of time" characterized by the path or the trajectory x(t) of the particle. He then extended the quantum superposition principle to apply for paths making use of the following basic distinction between classical and quantum probabilities.

Let $P(a|b)$ denote the conditional probability of an event a given that the event b occurred and similarly for $P(a|c)$ and $P(b|c)$. Then, classically one has the rule

$$P(a|c) = \sum_b P(a|b)\, P(b|c) \tag{2.7}$$

where the sum is over all events (or states) b that can occur between c and a. The quantum mechanical rule , however, is somewhat different. Here we have to work with a probability amplitude ϕ_{ab} which like $P(a|b)$ depends on two states and follows a relation similar to (2.7)

$$\phi_{ac} = \sum_b \phi_{ab}\, \phi_{bc} \tag{2.8}$$

where as before the sum is over all possible states b. The difference in these two situations is that ϕ is not a probability but amplitude such that $|\phi|^2$ is interpreted as a probability. Equation (2.8) is, in fact,

the quantum superposition law as extended to paths and is illustrated in Fig. 1.1 by the double slit experiment.

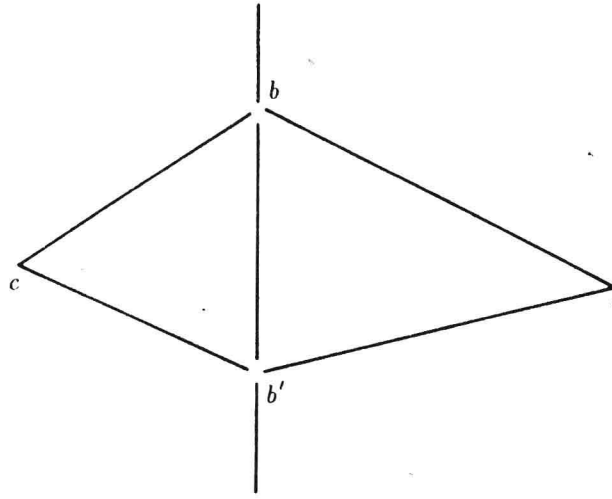

Fig. 1.1. Double slit experiment. *The electron at c can reach the point a on the screen either by taking the path cba or the path cb′a. The amplitude for the path cba is* $\phi_{abc} = \phi_{ab} \cdot \phi_{bc}$ *and that for the path cb′a is* $\phi_{ab'c} = \phi_{ab'} \cdot \phi_{b'c}$. *The total amplitude is thus* $\phi_{ac} = \phi_{abc} + \phi_{ab'c} = \phi_{ab}\phi_{bc} + \phi_{ab'}\phi_{b'c}$. *Here* ϕ_{ab} *is the amplitude for the path ab, etc.*

An electron passes through either (or both) of two slits on its way to the screen. The superposition rule would imply an interference pattern on the screen. Any attempt to verify through which slit the electron went will destroy the interference pattern. Feynman translated this in the Language of "sum over paths" by simply interpreting ϕ_{ab} to be the amplitude for "path" of the electron from b to a and used (2.8)

as a postulate to derive his path integral formulation. There is no assertion here that the particle followed definite paths with certain probabilities. One has to obtain the probability amplitudes for various paths and add them to obtain the total amplitude.

In analogy with the double slit experiment the basic quantity in Feynman formulation is the probability amplitude $K(x'',t'';x',t')$ for a non-relativistic particle to go from a space-time point (x',t') to (x'',t''). We must then sum over all intermediate possibilities, that is, our sum must include contributions from each trajectory connecting (x',t') to (x'',t''). If we denote the probability amplitude for a trajectory $x(t)$ $(x(t') = x'$ and $x(t'') = x'')$ by $\phi[x(t)]$, the total amplitude

$$K(x'',t'';x',t') = \sum_{(x(t))} \phi\,[x(t)] \,. \tag{2.9}$$

How do we obtain $\phi\,[x(t)]$? It is here that connection with Dirac's idea comes in. Dirac noticed that the transformation function K, taking particle from (x',t') to (x'',t''), more commonly known as the propagator is "analogous" to $\exp\,[iS/\hbar]$ where S is the solution of the HJ equation. As mentioned earlier, this is to be expected for small $\hbar$ (WKB approximation). However, Dirac also observed that $\exp\,[iS/\hbar]$ is also a reasonable approximation when the time interval over which K propagates tends to zero. Thus for the simple Lagrangian of a particle of mass m moving in a potential V(x) given by

$$L = \frac{1}{2} m \dot{x}^2 - V(x) \,, \tag{2.10}$$

Dirac propagator assumes the form

$$K(x,y;\varepsilon) = K(x,t+\varepsilon;y,t) = \frac{1}{A} \exp\left\{\frac{i\varepsilon}{\hbar}\left[\frac{m}{2}\left(\frac{x-y}{\varepsilon}\right)^2 - V(x)\right]\right\} \tag{2.11}$$

for small ε. A consequence of the superposition law is the integral equation

$$\psi(x,t+\varepsilon) = \int K(x,t+\varepsilon;y,t)\,\psi(y,t)\,dy \tag{2.12}$$

connecting the wave function at time t to the wave function at time $t+\varepsilon$. Summation over intermediate state b now corresponds to the particle position y at time t. If (2.11) is inserted in (2.12), one obtains