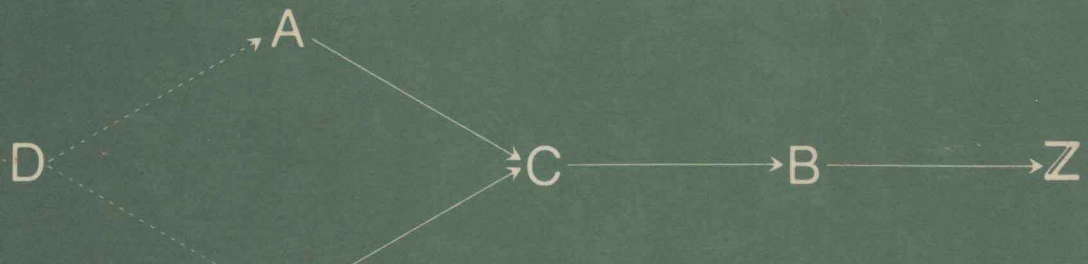
Semigroups and Their Applications

Simon M. Goberstein and
Peter M. Higgins (eds.)



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Semigroups and Their Applications

Proceedings of the International Conference "Algebraic Theory of Semigroups and Its Applications" held at the California State University, Chico, April 10-12, 1986

edited by

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Semigroups and Their Applications

PREFACE

Most papers published in this volume are based on lectures presented at the Chico Conference on Semigroups held on the Chico campus of the California State University on April 10-12, 1986. The conference was sponsored by the California State University, Chico in cooperation with the Engineering Computer Sciences Department of the Pacific Gas and Electric Company. The program included seven 50-minute addresses and seventeen 30-minute lectures. Speakers were invited by the organizing committee consisting of S. M. Goberstein and P. M. Higgins.

The purpose of the conference was to bring together some of the leading researchers in the area of semigroup theory for a discussion of major recent developments in the field. The algebraic theory of semigroups is growing so rapidly and new important results are being produced at such a rate that the need for another meeting was well-justified. It was hoped that the conference would help to disseminate new results more rapidly among those working in semigroups and related areas and that the exchange of ideas would stimulate research in the subject even further. These hopes were realized beyond all expectations. All talks presented at the conference were of high quality; together, they covered a vast array of topics: semigroup algebras, transformation semigroups, varieties and pseudovarieties of some classes of semigroups, global semigroup theory, various topics in the theory of inverse semigroups, semigroups presented by relations, and connections with languages and automata. An enlightening lecture presented at the conference by T. E. Hall about C. Ash's brilliant result that every finite monoid with commuting idempotents is a morphic image of a subsemigroup of a finite inverse semigroup stimulated J.-C. Birget, S. Margolis and J. Rhodes to extend Ash's ideas and to prove that every finite semigroup whose idempotents form a subsemigroup divides a finite orthodox semigroup. These results show that the so-called "Type II Conjecture" stated by J. Rhodes and B. Tilson in 1972 holds for finite monoids whose idempotents form a submonoid. B. Tilson wrote an article, specifically for this volume, clarifying the proof of the original result that led to the Type II Conjecture. This article is included in the Proceedings even though it was not presented at the conference.

The papers in this volume are written in a variety of forms. Some represent extensive surveys, others are written as regular research articles. A number of contributions are research announcements whose proofs should appear elsewhere. And several papers represent a mixture of all three categories of articles. There is no need to discuss here the contents of each contribution. The reader is advised to read them all.

This conference would never have taken place if it had not been generously financed by the Office of the Provost, the Graduate School and the College of Natural Sciences of the California State University, Chico and the Pacific Gas and Electric Company. The support of Provost G. R. Stairs and Associate Vice President for Research J. S. Morgan was especially vital. R. J. Bakke from the Graduate School worked hard trying to raise external funds to support the conference and B. L. Lundy from the College of Natural Sciences did a lot of work as the conference accountant. T. A. McCready, Chairman of the Department of Mathematics, constantly encouraged and supported me during the preparation of the conference and the editing of its Proceedings. A. Bowman, S. Jones and B. Stansbury, Mathematics Department secretaries, worked tirelessly typing various materials related to the conference as well as parts of this volume. The work and help of all these individuals is greatly appreciated.

Shortly after the first announcement about the conference had been mailed, my colleague and friend, P. M. Higgins, left Chico to take an appointment at Deakin University. In spite of the great distance between Chico and Geelong, Peter's interest in the conference never weakened. He arrived in Chico a week before the meeting and was eager to help with the final preparations. For all that and for his willingness to help me with editing some of the papers for this volume I owe him my gratitude.

Last but not least I am especially grateful to my wife Faina for her encouragement and understanding during the preparation of the meeting and the editing of this volume and for being such a gracious hostess to the conference participants.

Chico, California, December 26, 1986 Simon M. Goberstein

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SCIENTIFIC PROGRAM

Thursday, April 10

Morning session (Chairman - S.M. Goberstein):

Norman Reilly, Simon Fraser University (Canada)
'The lattice of varieties of completely regular semigroups'
(a 50-minute address)

Jorge Almeida, Universidade do Minho (Portugal)
'Implicit operations on certain classes of semigroups'

William Nico, California State University, Hayward
'Categorical extension theory - revisited'

Francis Pastijn, Marquette University
'Uniform chains'

Afternoon session (Chairman - J. Rhodes):

W.D. Munn, University of Glasgow (Scotland)
'A class of inverse semigroup rings' (a 50-minute address)

K. Johnston, College of Charleston
'Modular inverse semigroups'

T. Tamura, University of California, Davis
'On the recent study of power semigroups'

J.-E. Pin, Université Pierre et Marie Curie-Paris VI (France)
'Finite power semigroups: A survey'

J.A. Gerhard, University of Manitoba (Canada)
'Some free semigroups and free *-semigroups'

Friday, April 11

Morning session (Chairman - P.M. Higgins):

John Rhodes, University of California, Berkeley
'New techniques in global semigroup theory' (a 50-minute address)

Stuart Margolis, University of Nebraska-Lincoln
'Word problems for inverse semigroups'

J.C. Meakin, University of Nebraska-Lincoln
'The E-unitary problem for one-relator inverse monoids'

W. Lex, Technische Universität Clausthal (F.R.G.)
'Lattices of torsion theories for semi-automata'

Afternoon session (Chairman - R.J. Koch):

T.E. Hall, Monash University (Australia)
'Ash's theorem: any finite semigroup with commuting idempotents divides a finite inverse semigroup' (a 50-minute address)

Karl Byleen, Marquette University
'Inverse semigroups with countable universal semilattices'

J.-C. Birget, University of Nebraska-Lincoln
'Global theory of semigroups whose idempotents commute'

Mario Petrich, Simon Fraser University (Canada)
'Cayley theorems for semigroups' (a 50-minute colloquium address)

Saturday, April 12

Morning session (Chairman - J.M. Howie):

Peter Jones, Marquette University
'Basis properties, exchange properties and embeddings in idempotent-free semigroups' (a 50-minute address)

M.P. Drazin, Purdue University
'Units of inverse semigroup algebras'

K.D. Magill, SUNY at Buffalo
'The countability indices of certain transformation semigroups'

Matthew Gould, Vanderbilt University
'Cofinality of normal bands of groups'

Afternoon session (Chairman - W.D. Munn):

T.E. Hall, Monash University (Australia)
'The amalgamation bases of the class of finite inverse semigroups' (a 50-minute address)

Howard Straubing, Boston College
'Partially ordered finite monoids and a theorem of I. Simon'

J.M. Howie, University of St. Andrews (Scotland)
'Rank properties in semigroups of mappings'

TABLE OF CONTENTS

IMPLICIT OPERATIONS ON CERTAIN CLASSES OF SEMIGROUPS

Jorge Almeida* and Assis Azevedo*
Centro de Matemática
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Braga, Portugal

ABSTRACT. Varieties of algebras are characterized by identities, where an identity is a formal equality of two terms (i.e., operations defined by means of the underlying operations). Analogously, pseudovarieties of (finite) algebras are defined by pseudo-identities, these being formal equalities of so-called implicit operations (briefly , functions compatible with all homomorphisms). To further explore this analogy to yield results on finite algebras, it is necessary to obtain clear descriptions of implicit operations. This work is a contribution to this project in the area of semigroup theory. All unary implicit operations on semigroups are described, and the implicit operations on certain pseudovarieties of semigroups are given in terms of "generating" operations. The existence of some unusual implicit operations is established based on classical combinatorial theorems about words.

1. INTRODUCTION

This paper is concerned with a new way of looking at pseudovarieties of semigroups : via implicit operations. Roughly put, an implicit operation is a new operation which is preserved by all functions that preserve the old operations (i.e., homomorphisms). Reiterman [8] showed that implicit operations on finite algebras of a finite finitary type form a compact metric space in which the subset of finite composites of old operations is dense. He also showed that pseudovarieties are defined by pseudoidentities, i.e., by formal equalities of implicit operations, thus providing a suitable analog to the classical Birkhoff theorem on varieties.

We start here a systematic study of implicit operations on finite semigroups. Our first positive result is a full constructive description of unary implicit operations. This already allows us to show there is a vast unexplored world of implicit operations compared with what can be found in the literature on pseudovarieties (cf. Eilenberg [4] and Pin [7]).

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We then proceed to show there is apparently more and more complication as the arity n of implicit operations increases. For n=2,3, this is based on classical results on avoidable regularities in words on two and three-letter alphabets.

Finally, we restrict our attention to the class ZE of all finite semigroups in which idempotents are central. We show that implicit operations on finite groups and explicit operations suffice to obtain all implicit operations on ZE . This is based on a careful study of sequences of words depending on some simple algebraic and combinatorial lemmas.

For basic notation on semigroups and pseudovarieties, the reader is referred to Lallement [5].

2. BACKGROUND AND NOTATION

A class V of finite algebras of a given type is said to be a pseudovariety if it is closed under homomorphic images, subalgebras and finitary direct products. We say V is equational if there is a set Σ of identities such that V is the class of all finite algebras of the given type which satisfy all the identities in Σ .

EXAMPLE 2.1. The pseudovariety N of all finite nilpotent semigroups is not equational since N satisfies no nontrivial semigroup identities, i.e., the least equational pseudovariety containing N is the class S of all finite semigroups. Thus, identities do not suffice to define pseudovarieties.

Let C be any class of algebras of a given type. An n-ary implicit operation π on C associates with each $A \in C$ an n-ary operation $\pi_A : A^n \longrightarrow A$ in such a way that if $A,B \in C$ and $\varphi : A \longrightarrow B$ is a homomorphism then $\pi_B(\varphi(a_1),\ldots,\varphi(a_n)) = \varphi(\pi_A(a_1,\ldots,a_n))$ for all $a_1,\ldots,a_n \in A$. The class of all n-ary implicit operations on C is denoted by $\overline{\Omega}_n C$.

For $A \in C$ and $\pi,\rho \in \overline{\Omega}_n C$, we write $A \models \pi = \rho$ to mean $\pi_A = \rho_A$; we call $\pi = \rho$ a pseudoidentity for C and we then say A satisfies this pseudoidentity. If Σ is a set of pseudoidentities for C, $A \models \Sigma$ means A satisfies all the pseudoidentities in Σ and $[\![\Sigma]\!]$ denotes the class of all $A \in C$ such that $A \models \Sigma$.

If $t(x_1,\ldots,x_n)$ is a term of a given type τ in the variables $x_1,\ldots,x_n$, then t defines an n-ary implicit operation on any class C of algebras of type τ by letting t_A be the induced n-ary operation on $A \in C$ (where $t_A(a_1,\ldots,a_n)$ is obtained by "substituting" $a_i \in A$ for x_i $(i=1,\ldots,n)$). An implicit operation of this type is said to be explicit. The set of all n-ary explicit operations on C is represented by $\Omega_n C$.

In $\overline{\Omega}_n C$ it is possible to define a metric distance in the case - which we assume in the following - C is "essentially countable", i.e., up to isomorphism, there are only a countable number of algebras in C. This is the case, for instance, when C consists of finite algebras and the underlying type is finite and finitary. We will not describe here this distance function but rather the convergence of sequences in $\overline{\Omega}_n C$ since this is what will be used in the sequel. A sequence $(\rho_n)_n$ converges in $\overline{\Omega}_m C$ to π if, for every integer k, there is an integer n_k such that $A \models \pi = \rho_n$ for all $n \geq n_k$ and all $A \in C$ with $|A| \leq k$, where $|A|$ denotes the cardinality of A.

Next, we quote two important results of Reiterman [8] .

THEOREM 2.2. $\overline{\Omega}_n C$ is a compact metric space in which $\Omega_n C$ is dense.

THEOREM 2.3. Let $\underline{V}$ be a class of finite algebras of type τ . Then $\underline{V}$ is a pseudovariety if and only if there is a set Σ of pseudoidentities for the class of all finite algebras of type τ such that $\underline{V} = [\![\Sigma]\!]$.

In view of Theorem 2.3, it is natural to study pseudoidentities and implicit operations in order to achieve a better understanding of pseudovarieties.

From here on, we restrict our attention to classes of finite semigroups. For finite semigroups there is one more very natural implicit operation found in the literature besides the explicit operations : the idempotent unary operation x^ω . For an element s of a finite semigroup, the value s^ω of x^ω on s is the idempotent in the subsemigroup generated by s. There is, however, a need for more implicit operations in order to be able to define all pseudovarieties in terms of pseudoidentities.

EXAMPLE 2.4. Let $\underline{Ab}_p$ denote the class of all finite abelian p-groups, and let $\underline{Ab}^p$ denote the class of all finite abelian groups without elements of order p, where p is any prime. We claim $\underline{Ab}_p$ and $\underline{Ab}^p$ cannot be defined by pseudoidentities in which all implicit operations are composites of x^ω and explicit operations.

For, suppose $\underline{Ab}_p = [\![x^\omega = 1, \Sigma]\!]$ where Σ is a set of such pseudoidentities. (In general, $\pi=1$ is an abbreviation of $\pi(x_1,\ldots,x_n)y=y=y\pi(x_1,\ldots,x_n)$.) Then, every $\pi=\rho$ in Σ can be replaced by an identity of the form $v=1$ or $v=w$. Thus, we may assume Σ is a set of identities.

Let $[\Sigma]$ denote the class of all semigroups satisfying the identities in Σ . Of course, $[\Sigma]$ is a variety and the class $[\Sigma]^F$ of all finite members of $[\Sigma]$ is precisely $[\![\Sigma]\!]$. But $\mathbb{Z} \in [\Sigma]$ since $[\![\Sigma]\!]$ contains cyclic groups of arbitrarily large order and $[\Sigma]$ is a variety. Hence $\mathbb{Z}_q \in [\Sigma]^F \cap [\![x^\omega=1]\!] = \underline{Ab}_p$ for all q, contradicting the definition of $\underline{Ab}_p$. A similar argument works for $\underline{Ab}^p$.

3. UNARY IMPLICIT OPERATIONS

In a cyclic semigroup $< a ; a^n = a^{n+k} >$, we call n the index of a and k the period of a ; we also denote by K_a its maximal subgroup.

Let $\underline{S}$ denote the class of all finite semigroups.

LEMMA 3.1. Let $\pi \in \overline{\Omega}_1\underline{S}$ be such that $\pi_A(a) \notin K_a$ for some $A \in \underline{S}$ and $a \in A$. Then $\pi \in \Omega_1\underline{S}$.

PROOF. By Theorem 2.3, there is a sequence $(x^{\alpha_n})_n$ of words such that $\lim_{n\to\infty} x^{\alpha_n} = \pi$ in $\overline{\Omega}_1\underline{S}$. Now, if $\alpha_n \geq |A|$, then $\pi_A(a) \in K_a$, so that the set of exponents $\{\alpha_n : n = 1,2,\ldots\}$ must be bounded. Hence, there is a a constant sequence $(x^\alpha)_n$ converging to π in $\overline{\Omega}_1\underline{S}$, that is $\underline{S} \models \pi = x^\alpha$, whence $\pi \in \Omega_1\underline{S}$.

LEMMA 3.2. Let $\pi \in \overline{\Omega}_1\underline{S}$. Then, for $A \in \underline{S}$ and $a \in A$,
a) $\pi_A(a^s) = (\pi_A(a))^s$ for all positive integers s, and
b) $\pi_A(a^\omega) = a^\omega$.

PROOF. (a) Just note that $b \longmapsto b^s$ defines an endomorphism of any cyclic semigroup.
(b) This follows from (a) noting that there is an integer s such that $b^\omega = b^s$ for all $b \in A$.

Let $\underline{G}$ denote the class of all finite groups.

PROPOSITION 3.3 Let $\pi \in \overline{\Omega}_1\underline{S}$. Then, either $\pi \in \Omega_1\underline{S}$, or $\pi(x) = \pi(x^\omega x)$ so that π is completely determined by its restriction to $\underline{G}$.

PROOF. Suppose $\pi \notin \Omega_1\underline{S}$. Then, by Lemma 3.1, $\pi_A(a) \in K_a$ for all $A \in \underline{S}$ and $a \in A$. Thus, $\pi_A(a) = \pi_A(a)a^\omega$ since a^ω is the neutral element of K_a, whence $\pi_A(a) = \pi_A(a)\pi_A(a^\omega) = \pi_A(a^\omega a)$ applying Lemma 3.2 twice. Hence $\pi(x) = \pi(x^\omega x)$ as claimed.

We now study the unary implicit operations on $\underline{G}$. Let $\pi \in \overline{\Omega}_1\underline{G}$. Then, for $A \in \underline{G}$ and $a \in A$, $\pi_A(a) = a^{\alpha(n)}$ for some function $\alpha : \mathbb{N} \longrightarrow \mathbb{N}_0$ where n=ord a is the order of a (since $\pi_A(a) = \pi_{<a>}(a)$). The following Lemma gives the arithmetic conditions on such a function α which insure that the formula $\pi_A(a) = a^{\alpha(n)}$ defines an implicit operation on $\underline{G}$. We write $m|n$ in case m divides n. We denote by $\mathbb{N}$ the set of all positive integers and we let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

LEMMA 3.4. The following are equivalent for a function $\alpha : \mathbb{N} \longrightarrow \mathbb{N}_0$.
i) α defines an implicit operation on $\underline{G}$.
ii) $d|n$ implies $d|\alpha(n) - \alpha(d)$.

PROOF. (i) means, for every $A,B \in \underline{G}$, every homomorphism $h : A \to B$, and every $a \in A$, $h(\pi_A(a)) = \pi_B(h(a))$, i.e.,

$h(a^{\alpha(n)}) = (h(a))^{\alpha(d)}$ where n=ord a and d = ord h(a). Since h is a

homomorphism, $d|n$, whence $(h(a))^{\alpha(n)} = (h(a))^{\alpha(d)}$ if and only if $d|\ \alpha(n)-\alpha(d)$. This proves (ii) $\Rightarrow$ (i). For the converse, just use the above cyclic groups A and B of orders n and d and generators a and b, respectively, and the homomorphism $h : A \to B$ sending a to b.

Let $\mathbb{P}$ denote the set of all primes.