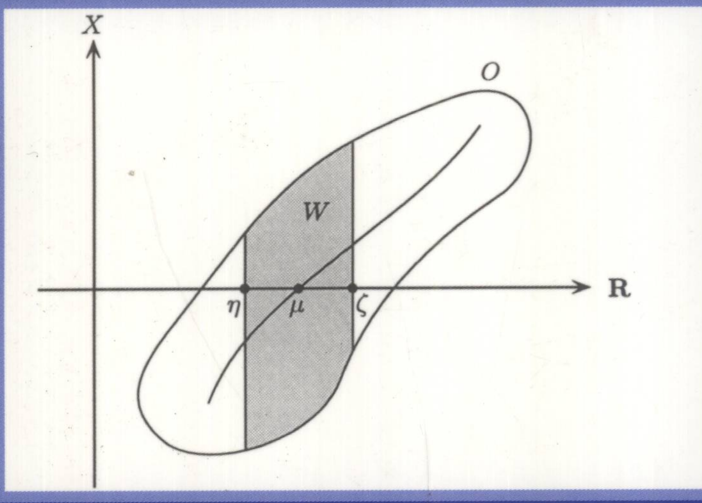
ROBERT F. BROWN

A TOPOLOGICAL INTRODUCTION TO NONLINEAR ANALYSIS

Second Edition



Birkhäuser

Robert F. Brown

A Topological Introduction to Nonlinear Analysis

Second Edition

Birkhäuser
Boston • Basel • Berlin

Robert F. Brown
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
USA

Library of Congress Cataloging-in-Publication Data

A CIP catalogue record for this book is available from the Library of Congress, Washington D.C., USA.

AMS Subject Classifications: 47H10, 55M20, 47H11, 34B15, 34C23, 34B60, 55M25, 47J25, 47J10, 34B25, 54H25, 34C25

ISBN 0-8176-3258-1

Printed on acid-free paper.

Birkhäuser

Printed in the United States of America. (SB)

9 8 7 6 5 4 3 2 1 SPIN 10936399

Birkhäuser is part of *Springer Science+Business Media*

www.birkhauser.com

To Brenda

Preface

Nonlinear analysis is a remarkable mixture of topology (of several different types), analysis (both "hard" and "soft") and applied mathematics. Mathematicians with a correspondingly wide variety of interests should become acquainted with this important, rapidly developing subject. But it's a BIG subject. You can feel it: just weigh in your hand Eberhard Zeidler's *Nonlinear Functional Analysis and its Applications I: Fixed Point Theorems* [13]. It's heavy, as a 900 page book must be. Yet this is no encyclopaedia; the preface accurately describes its "... very careful selection of material ..." And what you are holding is Part I of a five-part work.

So how do you get started learning nonlinear analysis? Zeidler's book has a first page, and some people are quite content to begin right there. For an alternative, the bibliography in [13], which is 42 pages long, contains exposition as well as research results: monographs that explain greater or lesser portions of the subject to a variety of audiences. In particular, [4] covers much of the material of Zeidler's book. Then what's different about the exposition in this book? My answer is in three parts: this book is (i) topological (ii) goal-oriented and (iii) a model of its subject. The next three little paragraphs explain what each of these means.

(i) As the title states, this is a *topological* book (though it's not a book of topology). I'm a topologist and, as I've studied nonlinear analysis I've became impressed by the extent to which the subject rests, in a strikingly simple and natural way, on basic topological ideas. These ideas come from general (point-set) topology, from metric space topology and, in the form of classical homology theory, from algebraic topology as well. It's possible to disguise, or even to replace to some extent, the substantial topological content of this subject, but that won't happen in this book. On the contrary, we'll make sure our analysis rests on a secure base of carefully-expounded topology.

(ii) The goal of this book has a name: the Krasnoselski–Rabinowitz bifurcation theorem. By the time you finish this book you will know what this beautiful result says, understand why it is true, and, through a single but very striking instance, get some idea of how it is applied. You can come to this book with little specific preparation. If you'll accept a few facts from elementary homology theory, this book is self-contained beyond the undergraduate real analysis level. Yet by the end of its relatively few pages you will see how, in the late 20th century (ca. 1970), we gained a new understanding of an 18th century mathematical model of a column collapsing under excessive weight.

(iii) Beyond its power and elegance, the Krasnoselski–Rabinowitz theorem has another virtue that made it irresistible as a topic for this book: the structure of its proof and this application is itself a model of the interplay of topological and analytical ideas that is characteristic of much of nonlinear analysis. The topological ingredients for the proof come from all the branches I mentioned: a separation theorem for compact topological spaces from general topology, Ascoli–Arzela theory from metric space topology, and the Leray–Schauder degree from algebraic topology. A key step in the proof is a calculation formula for Leray–Schauder degree which, in turn, depends on a substantial topic in functional ("soft") analysis: the spectral theory of compact linear operators on Banach spaces. The classical "hard" analysis comes into play once we have the relatively abstract bifurcation theorem and want to use it to study the ordinary differential equation problem that models column buckling.

As a curtain raiser to the relatively extensive discussions that lead us to the Krasnoselski–Rabinowitz theorem, I'll show you a simpler and more classical topological tool from the nonlinear analyst's toolbox: the Schauder fixed point theorem, along with a rather recent and easily understood application of it. This is also a model of nonlinear analysis: the topological topics of the Ascoli–Arzela theorem and fixed point theory are applied, with the help of some elementary but clever calculus, to investigate the equilibrium distribution of heat in a rod.

This book was born at a conference at the University of Montreal organized by Andrzej Granas in 1983 where the talks, especially those of Ronald Guenther, Roger Nussbaum and Paul Rabinowitz, made nonlinear analysis accessible. UCLA gave me the opportunity to communicate what I was learning about this subject, and to refine these notes, through specialized courses I taught in 1984, 1987 and 1992. The students and colleagues who attended these courses or talked to me about my plans for them helped me in many ways. I thank especially Joseph Bennish, Jerzy Dydak, Massimo Furi, Reiner Martin, and PierLuigi Zezza. The first time I taught about topology and nonlinear analysis, my late colleague Earl Coddington faithfully attended my lectures and didn't seem to think it was ridiculous for a topologist to try to present analysis from his own point of view. The fact that this book was written is a consequence of Earl's encouragement

About the Second Edition

Jean Mawhin's eloquent argument in [8] that much of nonlinear analysis could be illustrated in the context of the forced pendulum suggested some quite direct applications of the two main tools of the book, the Schauder fixed point theorem and the Leray–Schauder degree. In particular, the reader can now see a demonstration of the usefulness of the degree before being introduced to bifurcation theory. The book's contents have been restructured into three parts to reflect this change. The other significant addition consists of some background material from functional analysis and the theory of differential equations that now makes the book self-contained with regard to topics more advanced than undergraduate-level real analysis except for one subject. That subject is algebraic topology which, as in the first edition, receives only a bare-bones exposition. However, the task of putting flesh on those bones has become much easier with the recent publication of Hatcher's excellent algebraic topology text [7].

In 1998, Paul Rabinowitz was awarded the Birkhoff Prize by the American Mathematical Society and the Society for Industrial and Applied Mathematics, in large part because of the profound influence on nonlinear analysis of his remarkable theorem that is the goal of this book.

Robert Amodeo patiently guided me through various computer-related difficulties. I thank Ann Kostant, my editor at Birkhäuser, for suggesting that I prepare this revised and expanded edition.

Los Angeles, September 2003 Robert F. Brown

A Topological Introduction to Nonlinear Analysis

Second Edition

Contents

Part III Bifurcation Theory

Part IV Appendices

Part I

Fixed Point Existence Theory

1

The Topological Point of View

This book is about the topological approach to certain topics in analysis, but what does that really mean? Starting with the "epsilon–delta" parts of elementary calculus, analysis makes extensive use of topological ideas and techniques. Thus the issue is not whether analysis requires topology, but rather how central a role the topological material plays. Rather than attempt the hopeless task of defining precisely what I mean by the topological point of view in analysis, I'll illustrate it by outlining two proofs of a well-known theorem about the existence of solutions to ordinary differential equations. In the first proof, the key step is the construction of a sequence of approximate solutions whose limit is the required solution. In the second proof, a general topological theorem about the behavior of selfmaps of linear spaces implies the existence of the solution. The two proofs have several features in common, including their dependence on a substantial topological result, but I trust that even my (intentionally) very sketchy treatment will make it clear how basic the differences are in the ways that the two arguments reach the same conclusion. Here's the theorem.

Theorem 1.1. (Cauchy–Peano Existence Theorem) *Given a function* $f : \mathbf{R}^2 \to \mathbf{R}$ *which is continuous in a neighborhood of a point* $(x_0, y_0) \in \mathbf{R}^2$*, there exists* $\alpha > 0$ *and a solution to the initial-value problem*

$$y' = f(x, y) \qquad y(x_0) = y_0$$

on the interval $[x_0 - \alpha, x_0 + \alpha]$*. That is, there exists a continuous function* $\phi : [x_0 - \alpha, x_0 + \alpha] \to \mathbf{R}$ *such that* $\phi(x_0) = y_0$ *and* $\phi'(x) = f(x, \phi(x))$ *for all* x *in the interval.*

The two proofs produce the number α in the same way. Since f is continuous in a neighborhood of $(x_0, y_0) \in \mathbf{R}^2$, there exists $a > 0$ such that if $(x, y) \in \mathbf{R}^2$ with $|x - x_0| \leq a$ and $|y - y_0| \leq a$, then f is continuous at (x, y). Let Q be the square in the plane consisting of such points, that is,

$$Q = \{(x, y) \in \mathbf{R}^2 : |x - x_0| \leq a \quad \text{and} \quad |y - y_0| \leq a\}$$

and choose $M > 1$ such that $M \geq |f(x, y)|$ for all $(x, y) \in Q$. Then set $\alpha = \frac{a}{M}$. Notice how the definition of α depended on some familiar topology. A neighborhood means an open set and therefore the euclidean topology of the plane gives us an open disc about (x_0, y_0) on which f is continuous. We choose a small enough to fit the square Q inside the open disc. We know that the set of values $|f(x, y)|$ for $(x, y) \in Q$ is bounded, and therefore M exists, because Q is closed and bounded, that is, compact, so by a standard result its image under the continuous function $|f|$ is a compact subset of the line and therefore bounded.

Another feature the two proofs have in common is that they make use of the fact that the fundamental theorem of calculus gives us, as an equivalent form of the initial-value problem, the integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt.$$

That is, a function $\phi : [x_0 - \alpha, x_0 + \alpha] \to \mathbf{R}$ is a solution to the intial-value problem if and only if it is a solution to the integral equation.

The remaining common feature is that substantial topological result I referred to earlier: the Ascoli–Arzela theorem. I'll indicate in both proofs where and how this theorem is used, but in neither case is it necessary to state the result itself. However, I'll present a detailed discussion and proof of this theorem in the next chapter because the Ascoli–Arzela theorem will play a crucial role throughout the entire book.

1.1 Outline of the Approximation Proof

For each integer $n \geq 1$, choose $\delta_n > 0$ small enough so that $|x - \overline{x}| < \delta_n$ and $|y - \overline{y}| < \delta_n$ implies

$$|f(x, y) - f(\overline{x}, \overline{y})| < \frac{1}{n}.$$

Then choose points

$$x_0 - \alpha = x_{-k_n}^{(n)} < x_{-k_n+1}^{(n)} < \cdots < x_{-1}^{(n)} < x_0 < x_1^{(n)} < \cdots < x_{k'_n-1}^{(n)} < x_{k'_n}^{(n)}$$
$$= x_0 + \alpha$$

such that

$$\left| x_{j+1}^{(n)} - x_j^{(n)} \right| \leq \frac{\delta_n}{M}.$$

Define a piecewise-linear, continuous function $\phi_n : [x_0 - \alpha, x_0 + \alpha] \to \mathbf{R}$ in the following manner. See Figure 1. On the interval $[x_0, x_1^{(n)}]$, set $\phi_n(x_0) = y_0$ and let the slope of the line segment equal $f(x_0, y_0)$. On the interval $[x_1^{(n)}, x_2^{(n)}]$, the slope of the line segment is $f(x_1^{(n)}, y_1^{(n)})$, where $y_1^{(n)} = \phi_n(x_1^{(n)})$. Continue in this manner, moving to the right until you reach $x_{k'_n}^{(n)} = x_0 + \alpha$. Then define ϕ_n to the left of x_0 in a corresponding way. The function $\phi_n(t)$ is differentiable for all $t \neq x_j^{(n)}$, with

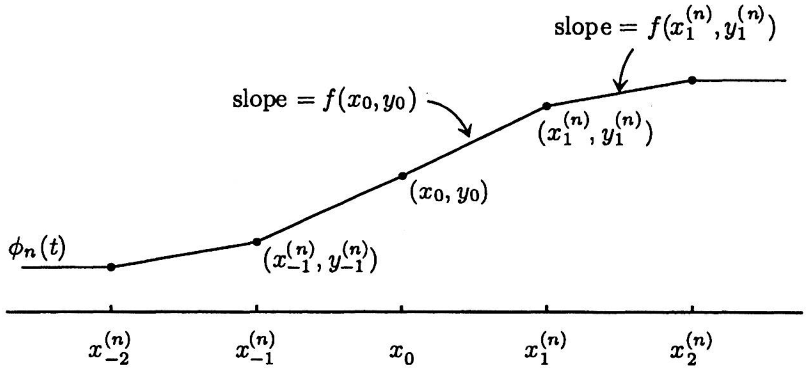


Figure 1.

$$\phi_n'(t) = f(x_j^{(n)}, \phi_n(x_j^{(n)}))$$

for $x_j^{(n)}$ an endpoint of the interval in which t lies.

The sequence of functions $\{\phi_n\}$ contains a subsequence that converges uniformly on the interval $[x_0 - \alpha, x_0 + \alpha]$. This is a consequence of the Ascoli–Arzela theorem that we will discuss in detail in the next chapter. For now, all we need to know is that it gives us a continuous function, the limit of the subsequence, and we denote that function by ϕ.

To complete the proof, we verify that ϕ is a solution to the initial-value problem. The argument consists of writing

$$\phi_n(x) = y_0 + \int_{x_0}^{x} f(t, \phi_n(t)) + \Delta_n(t)\, dt$$

where

$$\Delta_n(t) = \begin{cases} \phi_n'(t) - f(t, \phi_n(t)), & \text{if } t \neq x_j^{(n)} \\ 0, & \text{if } t = x_j^{(n)} \end{cases}$$

and noting that $|\Delta_n(t)| < \frac{1}{n}$ so that the limiting function ϕ does satisfy the equivalent integral equation. ■

1.2 Outline of the Topological Proof

Let C be the set of all real-valued functions that are continuous on the closed interval $[x_0 - \alpha, x_0 + \alpha]$. The set C inherits a linear space structure from the reals by defining, for $u, v \in C$ and $r \in \mathbf{R}$, the sum $(u + v)(x) = u(x) + v(x)$ and scalar product $ru(x) = r(u(x))$.

For this proof of the Cauchy–Peano theorem, a subset A of C is defined in the following way. We use the positive numbers M and α chosen before we began either proof. A function $u \in C$ is in A if it has the properties

$$(1)\ |u(x) - y_0| \leq a \quad \text{for all} \quad x \in [x_0 - \alpha, x_0 + \alpha],$$

$$(2)\ |u(x_1) - u(x_2)| \leq M|x_1 - x_2| \quad \text{for all} \quad x_1, x_2 \in [x_0 - \alpha, x_0 + \alpha].$$

The set A is a *convex* subset of C, that is, if $u, v \in A$ and $0 \leq t \leq 1$, then

$$tu + (1 - t)v \in A$$

also.

Define the *norm* $\|u\|$ of a function $u \in C$ by

$$\|u\| = \max\{|u(x)| : x_0 - \alpha \leq x \leq x_0 + \alpha\}.$$

A topology is defined on C by means of the metric d given by

$$d(u, v) = \|u - v\|$$

and the space C is called a *normed linear space*. The set A is closed in C with respect to this topology. If we apply the Ascoli–Arzela theorem in *this* proof, it tells us that the set A is compact.

Define a function $T : C \to C$ by letting

$$Tu(x) = y_0 + \int_{x_0}^{x} f(t, u(t))dt$$

for $u \in C$. We have seen that a solution to the integral equation equivalent to the initial-value problem is a function $\phi : [x_0 - \alpha, x_0 + \alpha] \to \mathbf{R}$ such that

$$\phi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))dt$$

for all $x \in [x_0 - \alpha, x_0 + \alpha]$. Since the right-hand side of this equation is $T\phi$, we see that ϕ has the property that $T\phi = \phi$. A point $u \in C$ is called a *fixed point* of $T : C \to C$ if $Tu = u$. Thus, to prove the Cauchy–Peano theorem, it is sufficient to prove that T has a fixed point.

The existence of a fixed point of T is a consequence of the following topological result.

Fixed Point Theorem. *If A is a compact, convex subset of a normed linear space X and $T : X \to X$ is a continuous function such that $T(A) \subseteq A$, then there exists $u \in A$ such that $Tu = u$.*

The continuity of the function T defined above follows from the continuity of f on Q. It can be shown that if $u \in A$, then Tu also satisfies the conditions (1) and (2) defining A and therefore $T(A) \subseteq A$. In this way, all the hypotheses of the fixed point theorem are verified and that completes the proof. ■

My purpose in presenting the two proofs of the Cauchy–Peano existence theorem was to illustrate two approaches, one of them characteristically topological; it was not to try to convince you that the topological approach is necessarily superior. In fact, the explicit construction of approximate solutions may well be quite useful. The topological method is important when there are no other, more concrete, alternatives available. We will see in the rest of the book that, in such cases, topology can offer us important insights into significant analytic problems.