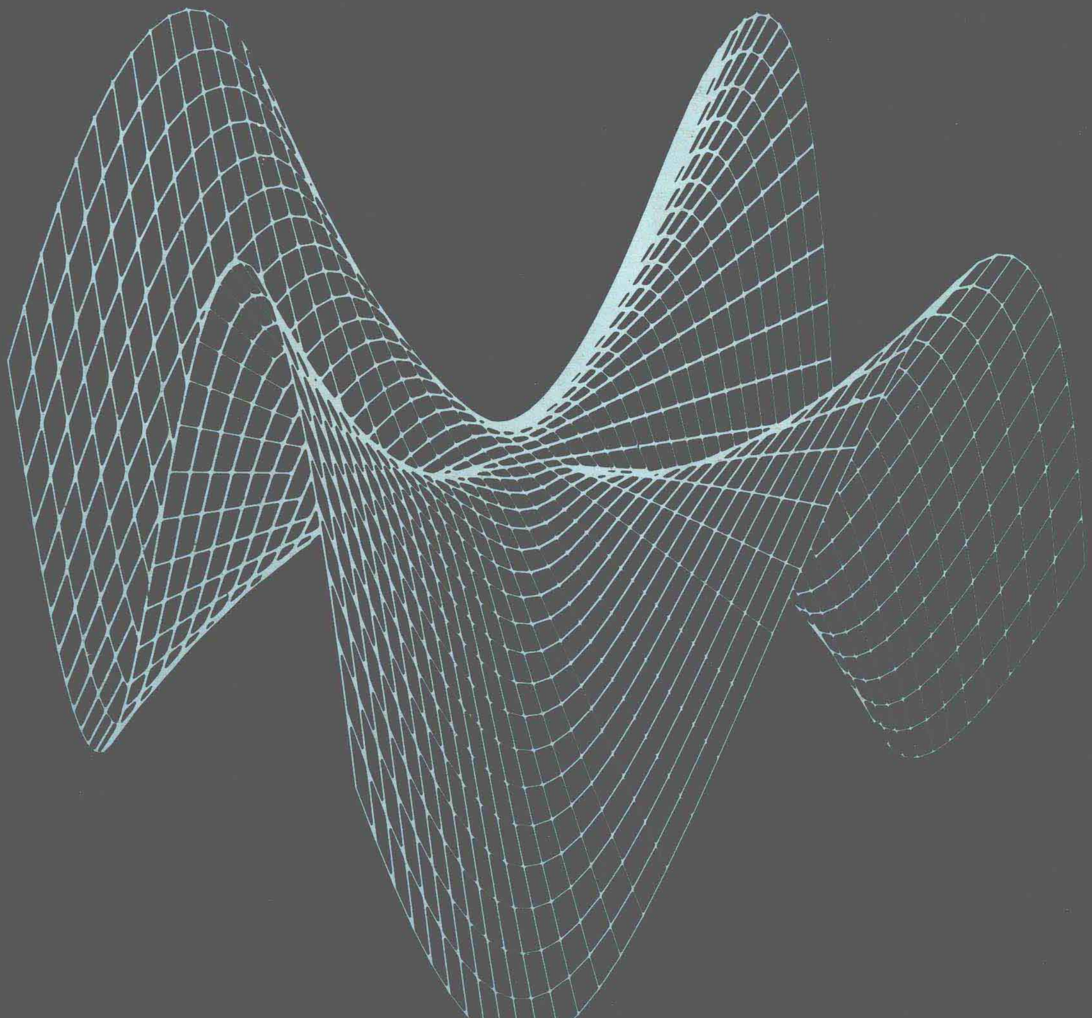
INTERMEDIATE CALCULUS

JAMES F. HURLEY



INTERMEDIATE CALCULUS

MULTIVARIABLE FUNCTIONS AND DIFFERENTIAL EQUATIONS WITH APPLICATIONS

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To Cecile, JoJo, and Gigi

PREFACE

This book is designed to introduce the calculus of functions of several variables and differential equations to students who have completed a one-year course in elementary calculus. Care has been taken to motivate each new topic prior to its study. This is usually done by relating it to notions the student has met previously or by showing that it is an appropriate tool for solving some naturally arising problem or for extending some basic idea.

Multivariable calculus is developed by searching for analogues of familiar results and techniques of elementary calculus. To make the analogies as sharp as possible, vector notation such as $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$ is used. This not only helps to bring out the essence of the mathematical ideas, but also formulates them in the notation most commonly used in the science and engineering courses that many of the students take concurrently. Considerable attention is given to geometric ideas in the early chapters. It is my belief that students who are thoroughly familiar with curves and surfaces in three dimensions are well equipped to set up the multiple, line, and surface integrals that occur later.

In discussing both multivariable calculus and differential equations, ideas and techniques of linear algebra have been used wherever they contribute to a clearer understanding of the material being presented. No previous experience with linear algebra is needed. Rather, an attempt has been made to integrate linear algebra into the presentation in much the same way that elementary calculus texts integrate analytic geometry. This approach can serve to introduce the main computational techniques to students who do not take a formal course in linear algebra. For students who have taken or are taking such a course, it can serve to illustrate the wide applicability of the subject.

Recommendations of the Committee on the Undergraduate Program in Mathematics (CUPM) of the Mathematical Association of America and of the National Research Council Committee on Applied Mathematics Training, in particular those urging realistic applications from a variety of fields in lower-division courses, have played a major role in structuring the book. A large number of examples and exercises involve applications of the mathematical ideas to other disciplines. The chapters on differential equations particularly emphasize mathematical modeling; they present applications not only to the physical sciences and engineering but also to fields such as psychology, ecology, demography, economics, criminology, and political science. No background in any of the applied fields is needed or assumed.

Numerous historical notes appear, both to give credit to those who have built the subject and to make the student aware of mathematics as a continually developing area of knowledge.

Differentiation is approached by way of the twentieth century notion of the total derivative. This is introduced heuristically as what ought to result from putting together in an appropriate way all the partial derivatives of a real valued function of several real variables. After a close look at the nature of the derivative in elementary calculus, the total derivative of f is defined from the point of view of local linear approximation of f. Early experience with this important concept illuminates the nature of differentiation and points the way to linear algebra, modern advanced calculus, and differential equations for students continuing in mathematics.

Double and triple integrals over rectangles and rectangular parallelepipeds are discussed first because of their simplicity and direct analogy to definite integrals over closed intervals. Fubini's Theorem is the central computational tool used in evaluating multiple integrals, and sections on polar, cylindrical, and spherical coordinate integrals lead ultimately to change of variables in multiple integrals. The treatment of vector integration includes the Theorems of Green, Gauss, and Stokes, as well as several classical applications to engineering and physics.

Chapter 8 presents first order and linear second order differential equations, with a review of complex numbers. The constant coefficient case is emphasized, but there is a section devoted to power series and numerical solutions. Chapter 9 considers n-th order linear equations and Chapter 10 treats first order systems. The Laplace transform is introduced as a means to solve initial value problems in both chapters. This material can be studied before the calculus chapters are begun. An independent chapter introduces infinite series, which are referred to in several places in the rest of the text.

I am grateful to a great many individuals and institutions who have contributed to the evolution and improvement of this work. Cecile N. Hurley not only supplied and assisted with numerous applied examples, but also patiently and faithfully typed the the entire manuscript from handwritten copy whose legibility often left much to be desired. Robert J. Weber showed me a number of interesting applications and referred me to others during many stimulating conversations and lectures on mathematical modeling. Professors Beverly Henderson of Cornell University, Robert Martin of the University of Pennsylvania, and Kenneth Bogart of Dartmouth University made encouraging and helpful reviews of an early version of the book. Professors Bogart, John Scheick of Ohio State University, and John Schiller of Temple University carefully read the semi-final manuscript, and their extraordinarily thorough reviews and suggestions helped eliminate many errors and significantly improve the exposition. Professor Philip Gillett of the University of Wisconsin Marathon Center checked the final manuscript and is responsible for the correction of other errors and for several additional improvements. (Remaining defects are, of course, solely my responsibility.) David Mohrman and Anthony Chiodo assisted in compiling and checking answers to the exercises. Joseph Hurley helped with final manuscript preparations.

I am also indebted to my department head, John V. Ryff, who permitted me to class-test two earlier versions of the manuscript. My classes of 1976–77 and 1977–78 contributed countless valuable suggestions while cheerfully coping with a textbook-in-the-making. To these classes, and to earlier ones at the University of Connecticut, the University of the Philippines, Ateneo de Manila University, De La Salle University, and the University of California campuses at Riverside and Los Angeles, I am indebted for enthusiastic reception of my ideas, for the many suggestions to put them into written form, and for the encouragement to see this project through to completion.

Over the past fifteen years, I have taught sophomore mathematics from texts authored by many mathematicians, and have consulted numerous others in the course of preparing lectures. All of these have helped me to shape my view of the material contained here, and their contribution is gratefully acknowledged. I have also benefitted during this period from grants under the Fulbright-Hays program and the National Science Foundation's Scientists and Engineers in Economic Development program, which afforded me the opportunity to teach students with a wide range of backgrounds. The influence of Louis Leithold, my own intermediate calculus instructor, will be evident to anyone familiar with Professor Leithold's exemplary texts. Finally, it is a pleasure to thank the editorial and production staffs of W.B. Saunders Company, whose professionalism and skill have been so helpful. In particular, Jay Freedman's contributions in both developmental and copy editing have significantly raised the quality of the book. The enthusiastic faith of Saunders' Mathematics Editor, Bill Karjane, in this project over the past three years has truly been the major contributing force to its publication.

JAMES F. HURLEY

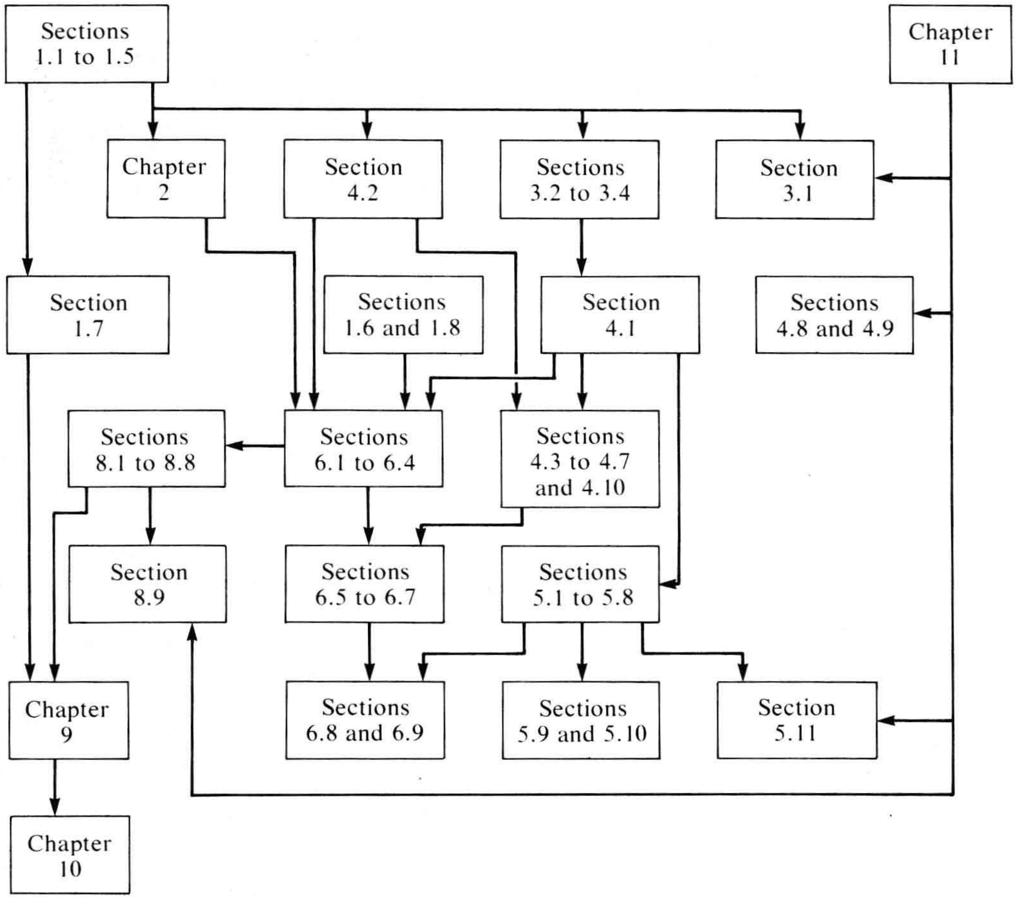
FOREWORD TO THE INSTRUCTOR

This text is written with flexibility in mind, and it can be used for a variety of courses enrolling students who have completed a year of calculus. Chapter 1 of the accompanying *Instructor's Manual* explores various possible courses that can be given from the text, and includes sample outlines. At the University of Connecticut, Chapters 1 through 5 (omitting Section 1.7) have been taught in the first semester and Chapters 6 through 10 (plus Section 1.7) in the second semester. Chapter 11 can replace parts of Chapters 1 and 2 for students with previous knowledge of vectors.

Each section normally says more that you can, or will want to, cover in the classroom. The *Instructor's Manual* contains suggestions about specific items that can be left for student reading. Sections to which more than one class period can be devoted include 1.8, 4.3, 4.5, 7.2, 7.6 to 7.8, 8.2, 8.8, 9.3, 10.2, and 10.4.

The flow chart on the following page presents section and chapter dependency in detail, and the *Instructor's Manual* discusses several alternatives to following the natural order of chapters. Note particularly that the text can be used for two separate sophomore courses: a 4- or 5-credit third semester (or fourth quarter) calculus course, and a 3- or 4-credit course in linear algebra and differential equations. The former would cover as much of Chapters 1 to 7 as time allows. To save time, Chapters 4 and 6 can be done together in the order 4.1 to 4.3, 6.1 to 6.5, 4.4, 4.5–6.6, 4.6–6.7, 4.7 to 4.10. Sections 6.8 and 6.9 can be omitted or integrated into Sections 5.5, 5.7, and 5.8. The differential equations course can be constructed from Sections 1.6 to 1.8, 4.2, 6.1 to 6.4, and Chapters 8, 9, and 10. Note also that Chapter 8 can be covered independently of linear algebra and at virtually any point of a year course taught from the book. Chapter 9 requires Sections 1.6 to 1.8 and 4.2 in addition to Chapter 8. After Section 10.1, Sections 6.1 to 6.4 are needed for full appreciation of Chapter 10.

Section 5.4 on polar coordinates and most of Chapter 11 on infinite sequences and series are written without proofs. The reason for this is that many elementary calculus classes will have discussed this material. For those students coming from such classes, these portions can serve as ready reference and review. You can, however, teach this material to students without previous background in it by filling in some elementary proofs.


Sections 1.1 to 1.5
Chapter 11
Chapter 2
Section 4.2
Sections 3.2 to 3.4
Section 3.1
Section 1.7
Sections 1.6 and 1.8
Section 4.1
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Sections 6.5 to 6.7
Sections 5.1 to 5.8
Chapter 9
Sections 6.8 and 6.9
Sections 5.9 and 5.10
Section 5.11
Chapter 10

FOREWORD TO THE STUDENT

This text is written for you to read and learn from. Since there is no such thing as a typical student, material has been included to meet the needs of a variety of students who enroll in courses following elementary calculus.

If you are a prospective mathematics major, then you will want to study not only the computational techniques and applications, but also the proofs and discussions of a more theoretical nature. On the other hand, if you are an engineering or physical, social, life, or management science major, then you will probably want to concentrate primarily on the computational techniques and illustrated uses of them in applied settings.

In any case, you will find the sections organized as follows. An introductory portion sets the scene, usually by reminding you of an important idea from your previous mathematical study which is related to the material to be presented. This is intended to help you to get your bearings and to understand *why* what follows is being done. Next come the definitions needed to give precise statements of the results of the section. These are important because they specify under exactly what circumstances the techniques can be used. If you take note of this, you can avoid the kind of frustration that often comes to students who try to apply machinery in contexts where it is inapplicable! Proofs are given in enough detail for you to follow step-by-step should you wish to. Your instructor will probably indicate to you the degree to which he thinks you should study these proofs. They are not required in order to do most exercises or follow the development of subsequent sections. In the References you can find sources containing omitted proofs that require more machinery or sublety than is appropriate to a book at this level. If you are ambitious, you can consult these references to obtain a more complete account of the subject. After the proofs, illustrations are given of how the material presented in the section is used in computational and/or applied problems. Study these carefully, because most of the exercises are based on ideas presented in these examples.

The exercises themselves are *essential* to mastering the material. Don't kid yourself—anybody who has passed an elementary calculus course can *follow* the solutions to the examples. But that is NOT the same as learning to *do* them! The latter ability can come only as a result of working the exercises. In many sections, you will find exercises that ask you to apply the techniques of the section to other mathematical and nonmathematical contexts. It is important to become adept at this, since much of the use given the material of this book lies in such applications.

Some exercises, generally toward the end of each list, are not just straightforward repetitions of the examples. Such exercises are designed to enable you to dig beneath the surface, to think about the material more deeply, and hence to improve your grasp of it. Hints are provided in many cases to help you get started. Many of these exercises are not proofs, and if you really want to *thoroughly* master intermediate calculus, you should look for such exercises and try them whether or not your instructor assigns them.

Answers to most odd-numbered computational problems appear at the end of the book. Every attempt has been made to ensure their accuracy, but they were all done and checked by human beings, many by students like you. If your answer differs from the one given, treat that as you would a discrepancy in answers with a classmate. Check your work, but don't assume in advance that the book's answers are infallible.

Definitions, lemmas, theorems, propositions (*i.e.,* little theorems), collaries, and examples are numbered sequentially within each section with a double number. Thus, Example 4.3 is the third numbered item in Section 4. A reference to Theorem 7.4 is to the fourth numbered item in the seventh section of the chapter being read. The presence of a third number in a reference tells which chapter (other than the one being read) contains it. Thus, Theorem 4.5.3 refers to the third numbered item in Section 5 of Chapter 4. Terms being defined are printed in **boldface** for emphasis. Proofs end with QED, which is the abbreviation for the Latin phrase *quod erat demonstrandum,* meaning *which was to be proved.* Until the nineteenth century, most mathematics was written in Latin, and this particular phrase is the Latin translation of the Greek words with which Euclid ended his proofs more than 2000 years ago, ὅπερ ἔδει δεῖξαι.

Finally, a word about how to read a mathematics book. In order to *really* understand, and learn, what you are reading, it is necessary to work through the development actively. That is, you should read with a pencil in your hand and a pad of paper at your side. Whenever the text claims that something follows from something else, the proper thing to do is to verify this for yourself by performing whatever algebraic calculation may be needed to derive the expression in question. **DON'T** make the two common mistakes of (a) simply taking the author's word for everything without trying to understand why it is true or (b) giving up in despair the first time you don't instantaneously see in your head how a particular expression results from another. The book has been written so that any steps not explicitly shown can be filled in by performing straightforward algebraic calculations. Doing such calculations will help you understand the material better and build confidence.

CONTENTS

1 VECTOR GEOMETRY AND ALGEBRA

0 INTRODUCTION

So far in your study of calculus, the main objects of discussion have been *real valued functions f of one real variable x*. For such functions we will use the notation

$$f : \boldsymbol{R} \to \boldsymbol{R} \quad \text{given by} \quad y = f(x)$$

where x is a real number in the domain D of f. (We generally will not write $f : D \to \boldsymbol{R}$ unless there is some need to emphasize the nature of the exact domain of f.) You have learned about graphs, limits, continuity, and integrals of these functions. In this text you will learn about the same concepts for *vector valued functions of several real variables*. These are functions $\boldsymbol{F}$ defined on domains D contained in some *n-dimensional Cartesian* (or *Euclidean*) *space* $\boldsymbol{R}^n$ and which take values in some space $\boldsymbol{R}^m$ (where m may or may not be the same as n). Our notation for such functions will be

$$\boldsymbol{F} : \boldsymbol{R}^n \to \boldsymbol{R}^m,$$

where $\boldsymbol{R}^1$ will denote the familiar set $\boldsymbol{R}$ of real numbers. (Again this means that the domain of $\boldsymbol{F}$ is some set $D \subseteq \boldsymbol{R}^n$ which may or may not be all of $\boldsymbol{R}^n$ and which we will usually not specify further.)

As you might imagine, a prerequisite to the study of functions defined on $\boldsymbol{R}^n$ is a working familiarity with such spaces. The goal of Chapter 1, then, is to develop that familiarity.

The first section introduces $\boldsymbol{R}^n$ to you via the real line $\boldsymbol{R}^1$, the Cartesian plane $\boldsymbol{R}^2$ (which should be old friends of yours by now), and Euclidean three-space $\boldsymbol{R}^3$, which in many ways is just $\boldsymbol{R}^2$ with a new perpendicular axis drawn to it.

In Sections 2 and 5, useful tools are developed for studying the *geometry* of $\boldsymbol{R}^n$, which is the main business of Sections 3, 4, and 5. The last three sections illustrate how certain geometric questions can be answered with purely algebraic tools. At the same time the questions will motivate an entirely new area of study, *linear algebra*. One of the themes of this book will be the repeated return to linear algebra as a source of methods to solve the problems of geometry and calculus which arise as we proceed.

1 THE SPACE R^n

The formal algebraic study of vectors goes back to the great nineteenth century Irish mathematician and physicist William Rowan Hamilton (1805–1865) and the German philosopher and mathematician Hermann G. Grassmann (1809–1877). But much earlier Sir Isaac Newton (1642–1727), one of the co-inventors of modern calculus, in his laws of motion dealt informally with geometric vectors as objects which possessed both length and direction. And in studying velocity, the ancient Greeks seem to have used vectors and even found resultant vectors by means of the parallelogram law. If you have studied physics, then you have no doubt dealt with vectors as geometric quantities with length and direction, and perhaps also as algebraic quantities. It is the latter point of view which goes back to Hamilton and Grassmann and which represents a profound contribution in the modern development of mathematics, science, and engineering. The description of vectors algebraically is our goal in this section.

Let us start with the geometric idea. A vector is a quantity which has both *length* and *direction*. Suppose we look at $\boldsymbol{R}=\boldsymbol{R}^1$ and ask how we might represent such a quantity. In $\boldsymbol{R}^1$ there are only two directions—positive (toward the right) and negative (toward the left)—and the assignment of coordinates to $\boldsymbol{R}^1$ affords a natural measurement of length. Thus, a vector three units long directed toward the right is quite simply described by (+3), while a vector of the same length directed to the left can be described by (−3). Notice that the length of the vector is simply its absolute value:

$$|(+3)|=3, \qquad |(-3)|=3.$$

In general, a vector $\vec{v}$ represented by (x) has length $|(x)|=\sqrt{x^2}$. This simple observation is the basis for our subsequent notation for the length of a vector.

Something else you may notice is that this description of vectors by real numbers carries with it no information about where a vector begins (or ends). Thus in Figure 1.1, (+2) describes both $\vec{v}_1$ and $\vec{v}_2$. While this may seem to be a

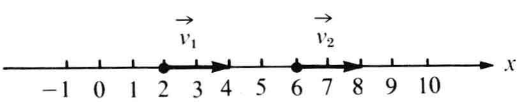

FIGURE 1.1

flaw in our representation scheme, it really is not. For in physics (and elsewhere), geometric vectors with the same length and same direction are identical, regardless of whether their initial points coincide. For this reason, the term "free vector" is often used to emphasize that one is free to draw a vector from any initial point one likes.

A representation of the vector drawn from a to b is $(b-a)$, since $b-a$ gives the directed distance from a to b (positive if $b>a$, negative if $b<a$, 0 if $b=a$). See Figure 1.2.

FIGURE 1.2

The resultant of two vectors in $\boldsymbol{R}^1$ is easily computed by algebraic addition. The equation

$$(+2)+(-3)=(-1)$$

is readily seen to say that the resultant of a vector of length two pointing to the right and a vector of length three pointing to the left is a vector of length one pointing to the left. Moreover, comparison of the space taken up by the equation and the statement in words suggests that the algebraic description we have developed is a valuable simplification.

Finally, notice that we can multiply vectors by real numbers in a natural way. The vector $a\vec{v}$ will be $\vec{0}$ if $a=0$, and otherwise will point in the same direction as $\vec{v}$ if $a>0$ or in the opposite direction from $\vec{v}$ if $a<0$. Its length will be $|a|$ times the length of $\vec{v}$. See Figure 1.3.

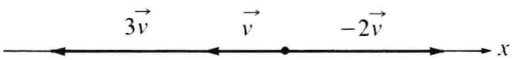

FIGURE 1.3

What about vectors in the plane $\boldsymbol{R}^2$? If we regard vectors as objects with length and direction, and identify those which have the same length and direction (no matter where their initial point happens to be), then how can we represent them algebraically? Since in $\boldsymbol{R}^1$ single real numbers gave us an adequate algebraic representation, we expect that in $\boldsymbol{R}^2$ we can use *pairs* of real numbers to represent vectors. Let's see how.

Since we are concerned only with length and direction, we can translate any given vector $\vec{v}$ parallel to itself until its beginning point is at the origin (Figure 1.4), and then use the coordinates (a, b) of its endpoint to represent $\vec{v}$. We know how to measure distances in $\boldsymbol{R}^2$ using the distance formula, so we see that the length of $\vec{v}$ is given by

$$|\vec{v}|=|(a, b)|=\sqrt{(a-0)^2+(b-0)^2}=\sqrt{a^2+b^2}.$$

Note the similarity between this and the formula for length of a vector in $\boldsymbol{R}^1$.

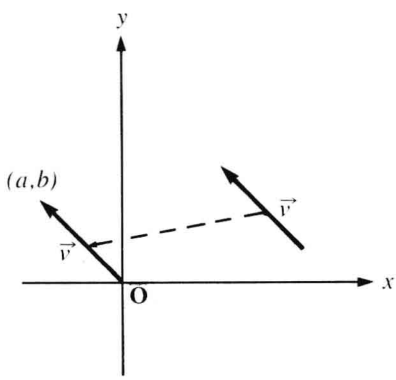

FIGURE 1.4

The analogy with one-dimensional vectors suggests that we should add vectors algebraically by adding corresponding coordinates of the representing ordered pairs, i.e.,

(*) $$(a, b)+(c, d)=(a+c, b+d).$$