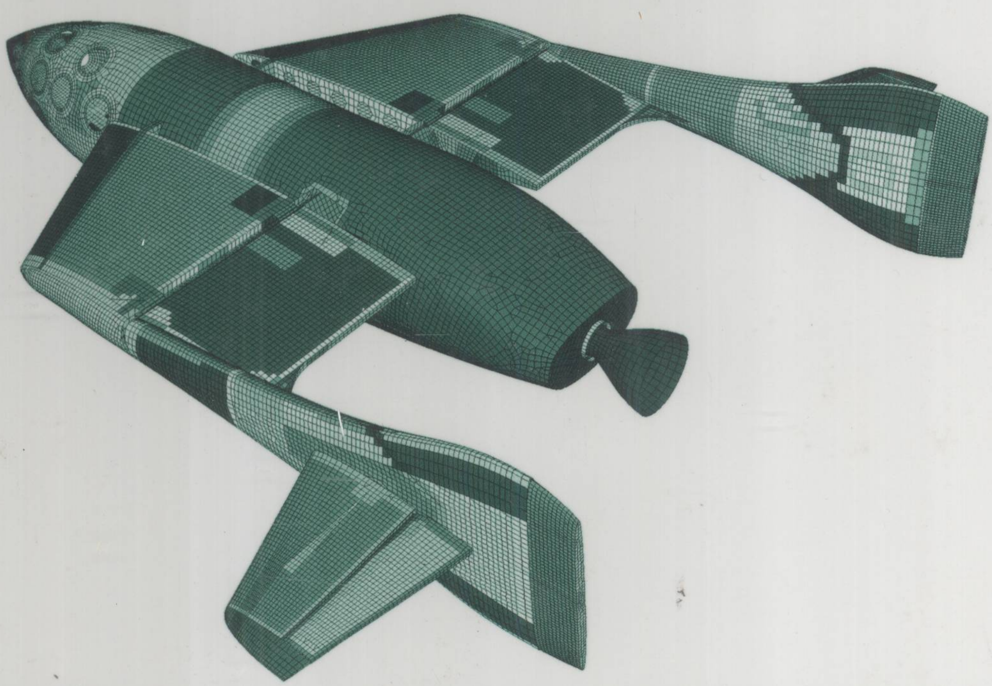
Applied Calculus of Variations for Engineers



Louis Komzsik

CRC Press
Taylor & Francis Group

Applied Calculus of Variations for Engineers

Louis Komzsik

CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business

CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

Printed in the United States of America on acid-free paper
10 9 8 7 6 5 4 3 2 1

International Standard Book Number-13: 978-1-4200-8662-1 (Hardcover)

Library of Congress Cataloging-in-Publication Data

Komzsik, Louis.
Applied calculus of variations for engineers / Louis Komzsik.
p. cm.
Includes bibliographical references and index.
ISBN 978-1-4200-8662-1 (hardcover : alk. paper)
1. Calculus of variations. 2. Engineering mathematics. I. Title.

TA347.C3K66 2009
620.001'51564--dc22 2008042179

Visit the Taylor & Francis Web site at
http://www.taylorandfrancis.com

and the CRC Press Web site at
http://www.crcpress.com

To my daughter, Stella

Acknowledgments

I am indebted to Professor Emeritus Bajcsay Pál of the Technical University of Budapest, my alma mater. His superior lectures about four decades ago founded my original interest in calculus of variations.

I thank my coworkers, Dr. Leonard Hoffnung, for his meticulous verification of the derivations, and Mr. Chris Mehling, for his very diligent proofreading. Their corrections and comments greatly contributed to the quality of the book.

I appreciate the thorough review of Professor Dr. John Brauer of the Milwaukee School of Engineering. His recommendations were instrumental in improving the clarity of some topics and the overall presentation.

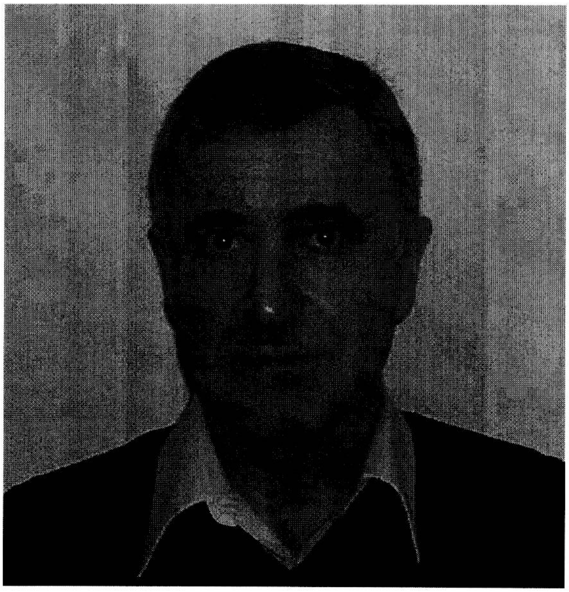
Special thanks are due to Nora Konopka, publisher of Taylor & Francis books, for believing in the importance of the topic and the timeliness of a new approach. I also appreciate the help of Ashley Gasque, editorial assistant, Amy Blalock and Stephanie Morkert, project coordinators, and the contributions of Michele Dimont, project editor.

I am grateful for the courtesy of Mojave Aerospace Ventures, LLC, and Quartus Engineering Incorporated for the model in the cover art. The model depicts a boosting configuration of SpaceShipOne, a Paul G. Allen project.

Louis Komzsik

About the Author

Louis Komzsik is a graduate of the Technical University of Budapest, Hungary, with a doctorate in mechanical engineering, and has worked in the industry for 35 years. He is currently the chief numerical analyst in the Office of Architecture and Technology at Siemens PLM Software.



Dr. Komzsik is the author of the *NASTRAN Numerical Methods Handbook* first published by MSC in 1987. His book, *The Lanczos Method*, published by SIAM, has also been translated into Japanese, Korean and Hungarian. His book, *Computational Techniques of Finite Element Analysis*, published by CRC Press, is in its second print, and his *Approximation Techniques for Engineers* was published by Taylor & Francis in 2006.

Preface

The topic of this book has a long history. Its fundamentals were laid down by icons of mathematics like Euler and Lagrange. It was once heralded as the panacea for all engineering optimization problems by suggesting that all one needs to do was to apply the Euler-Lagrange equation form and solve the resulting differential equation.

This, as most all encompassing solutions, turned out to be not always true and the resulting differential equations are not necessarily easy to solve. On the other hand, many of the differential equations commonly used by engineers today are derived from a variational problem. Hence, it is important and useful for engineers to delve into this topic.

The book is organized into two parts: theoretical foundation and engineering applications. The first part starts with the statement of the fundamental variational problem and its solution via the Euler-Lagrange equation. This is followed by the gradual extension to variational problems subject to constraints, containing functions of multiple variables and functionals with higher order derivatives. It continues with the inverse problem of variational calculus, when the origin is in the differential equation form and the corresponding variational problem is sought. The first part concludes with the direct solution techniques of variational problems, such as the Ritz, Galerkin and Kantorovich methods.

With the emphasis on applications, the second part starts with a detailed discussion of the geodesic concept of differential geometry and its extensions to higher order spaces. The computational geometry chapter covers the variational origin of natural splines and the variational formulation of B-splines under various constraints.

The final two chapters focus on analytic and computational mechanics. Topics of the first include the variational form and subsequent solution of several classical mechanical problems using Hamilton's principle. The last chapter discusses generalized coordinates and Lagrange's equations of motion. Some fundamental applications of elasticity, heat conduction and fluid mechanics as well as their computational technology conclude the book.

Contents

I Mathematical foundation 1

1 The foundations of calculus of variations 3

1.1 The fundamental problem and lemma of calculus of variations 3
1.2 The Legendre test 7
1.3 The Euler-Lagrange differential equation 9
1.4 Application: Minimal path problems 11
1.4.1 Shortest curve between two points 12
1.4.2 The brachistochrone problem 14
1.4.3 Fermat's principle 18
1.4.4 Particle moving in the gravitational field 20
1.5 Open boundary variational problems 21

2 Constrained variational problems 25

2.1 Algebraic boundary conditions 25
2.2 Lagrange's solution 27
2.3 Application: Iso-perimetric problems 29
2.3.1 Maximal area under curve with given length 29
2.3.2 Optimal shape of curve of given length under gravity 31
2.4 Closed-loop integrals 35

3 Multivariate functionals 37

3.1 Functionals with several functions 37
3.2 Variational problems in parametric form 38
3.3 Functionals with two independent variables 39
3.4 Application: Minimal surfaces 40
3.4.1 Minimal surfaces of revolution 43
3.5 Functionals with three independent variables 44

4 Higher order derivatives 49

4.1 The Euler-Poisson equation 49
4.2 The Euler-Poisson system of equations 51
4.3 Algebraic constraints on the derivative 52
4.4 Application: Linearization of second order problems 54

5 The inverse problem of the calculus of variations **57**
5.1 The variational form of Poisson's equation 58
5.2 The variational form of eigenvalue problems 59
5.2.1 Orthogonal eigensolutions 61
5.3 Application: Sturm-Liouville problems 62
5.3.1 Legendre's equation and polynomials 64

6 Direct methods of calculus of variations **69**
6.1 Euler's method 69
6.2 Ritz method 71
6.2.1 Application: Solution of Poisson's equation 75
6.3 Galerkin's method 76
6.4 Kantorovich's method 78

II Engineering applications **85**

7 Differential geometry **87**
7.1 The geodesic problem 87
7.1.1 Geodesics of a sphere 89
7.2 A system of differential equations for geodesic curves 90
7.2.1 Geodesics of surfaces of revolution 92
7.3 Geodesic curvature 95
7.3.1 Geodesic curvature of helix 97
7.4 Generalization of the geodesic concept 98

8 Computational geometry **101**
8.1 Natural splines 101
8.2 B-spline approximation 104
8.3 B-splines with point constraints 109
8.4 B-splines with tangent constraints 112
8.5 Generalization to higher dimensions 115

9 Analytic mechanics **119**
9.1 Hamilton's principle for mechanical systems 119
9.2 Elastic string vibrations 120
9.3 The elastic membrane 125
9.3.1 Nonzero boundary conditions 130
9.4 Bending of a beam under its own weight 132

10 Computational mechanics **139**
10.1 Three-dimensional elasticity 139
10.2 Lagrange's equations of motion 142
10.2.1 Hamilton's canonical equations 146
10.3 Heat conduction 148
10.4 Fluid mechanics 150
10.5 Computational techniques 153

10.5.1 Discretization of continua 153
10.5.2 Computation of basis functions 155

Closing Remarks **159**

Notation **161**

List of Tables **163**

List of Figures **165**

References **167**

Index **169**

Part I

Mathematical foundation

1

The foundations of calculus of variations

The problem of the calculus of variations evolves from the analysis of functions. In the analysis of functions the focus is on the relation between two sets of numbers, the independent (x) and the dependent (y) set. The function f creates a one-to-one correspondence between these two sets, denoted as

$$y = f(x).$$

The generalization of this concept is based on allowing the two sets not being restricted to be real numbers and to be functions themselves. The relationship between these sets is now called a functional. The topic of the calculus of variations is to find extrema of functionals, most commonly formulated in the form of an integral.

1.1 The fundamental problem and lemma of calculus of variations

The fundamental problem of the calculus of variations is to find the extremum (maximum or minimum) of the functional

$$I(y) = \int_{x_0}^{x_1} f(x, y, y')dx,$$

where the solution satisfies the boundary conditions

$$y(x_0) = y_0$$

and

$$y(x_1) = y_1.$$

This problem may be generalized to the cases when higher derivatives or multiple functions are given and will be discussed in Chapters 3 and 4, respectively. These problems may also be extended with constraints, the topic of Chapter 2.

A solution process may be arrived at with the following logic. Let us assume that there exists such a solution $y(x)$ for the above problem that satisfies the boundary conditions and produces the extremum of the functional. Furthermore, we assume that it is twice differentiable. In order to prove that this function results in an extremum, we need to prove that any alternative function does not attain the extremum.

We introduce an alternative solution function of the form

$$Y(x) = y(x) + \epsilon\eta(x),$$

where $\eta(x)$ is an arbitrary auxiliary function of x, that is also twice differentiable and vanishes at the boundary:

$$\eta(x_0) = \eta(x_1) = 0.$$

In consequence the following is also true:

$$Y(x_0) = y(x_0) = y_0$$

and

$$Y(x_1) = y(x_1) = y_1.$$

A typical relationship between these functions is shown in Figure 1.1 where the function is represented by the solid line and the alternative function by the dotted line. The dashed line represents the arbitrary auxiliary function.

Since the alternative function $Y(x)$ also satisfies the boundary conditions of the functional, we may substitute into the variational problem.

$$I(\epsilon) = \int_{x_0}^{x_1} f(x, Y, Y')dx.$$

where

$$Y'(x) = y'(x) + \epsilon\eta'(x).$$

The new functional in terms of ϵ is identical with the original in the case when $\epsilon = 0$ and has its extremum when

$$\frac{\partial I(\epsilon)}{\partial \epsilon}|_{\epsilon=0} = 0.$$

Executing the derivation and taking the derivative into the integral, since the limits are fixed, with the chain rule we obtain

$$\frac{\partial I(\epsilon)}{\partial \epsilon} = \int_{x_0}^{x_1} (\frac{\partial f}{\partial Y}\frac{dY}{d\epsilon} + \frac{\partial f}{\partial Y'}\frac{dY'}{d\epsilon})dx.$$

Clearly

$$\frac{dY}{d\epsilon} = \eta(x),$$

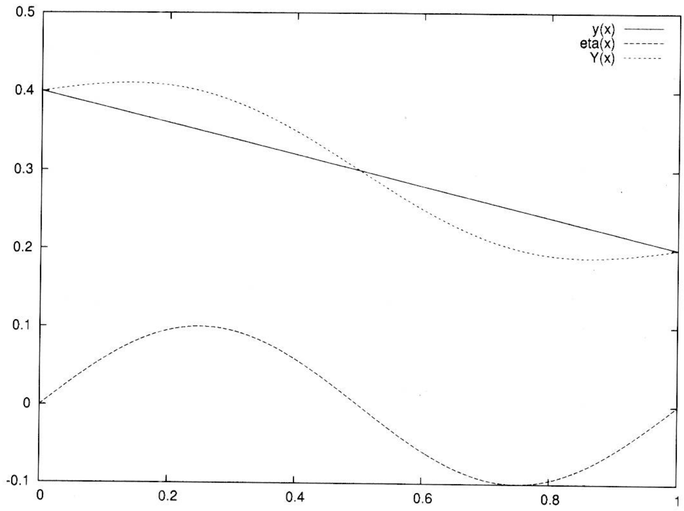


FIGURE 1.1 Alternative solutions example

and

$$\frac{dY'}{d\epsilon} = \eta'(x),$$

resulting in

$$\frac{\partial I(\epsilon)}{\partial \epsilon} = \int_{x_0}^{x_1} (\frac{\partial f}{\partial Y}\eta(x) + \frac{\partial f}{\partial Y'}\eta'(x))dx.$$

Integrating the second term by parts yields

$$\int_{x_0}^{x_1} (\frac{\partial f}{\partial Y'}\eta'(x))dx = \frac{\partial f}{\partial Y'}\eta(x)|_{x_0}^{x_1} - \int_{x_0}^{x_1} (\frac{d}{dx}\frac{\partial f}{\partial Y'})\eta(x)dx.$$

Due to the boundary conditions, the first term vanishes. With substitution and factoring the auxiliary function, the problem becomes

$$\frac{\partial I(\epsilon)}{\partial \epsilon} = \int_{x_0}^{x_1} (\frac{\partial f}{\partial Y} - \frac{d}{dx}\frac{\partial f}{\partial Y'})\eta(x)dx.$$

The extremum is achieved when $\epsilon = 0$ as stated above, hence

$$\frac{\partial I(\epsilon)}{\partial \epsilon}|_{\epsilon=0} = \int_{x_0}^{x_1} (\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'})\eta(x)dx.$$

Let us now consider the following integral:

$$\int_{x_0}^{x_1} \eta(x)F(x)dx,$$

where $x_0 \leq x \leq x_1$ and $F(x)$ is continuous, while $\eta(x)$ is continuously differentiable, satisfying

$$\eta(x_0) = \eta(x_1) = 0.$$

The fundamental lemma of calculus of variations states that if for all such $\eta(x)$

$$\int_{x_0}^{x_1} \eta(x)F(x)dx = 0,$$

then

$$F(x) = 0$$

in the whole interval.

The following proof by contradiction is from [13]. Let us assume that there exists at least one such location $x_0 \leq \zeta \leq x_1$ where $F(x)$ is not zero, for example

$$F(\zeta) > 0.$$

By the condition of continuity of $F(x)$ there must be a neighborhood of

$$\zeta - h \leq \zeta \leq \zeta + h$$

where $F(x) > 0$. In this case, however, the integral becomes

$$\int_{x_0}^{x_1} \eta(x)F(x)dx > 0,$$

for the right choice of $\eta(x)$, which contradicts the original assumption. Hence the statement of the lemma must be true.

Applying the lemma to this case results in the **Euler-Lagrange differential equation** specifying the extremum

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0.$$

1.2 The Legendre test

The Euler-Lagrange differential equation just introduced represents a necessary, but not sufficient, condition for the solution of the fundamental variational problem.

The alternative functional of

$$I(\epsilon) = \int_{x_0}^{x_1} f(x, Y, Y')dx,$$

may be expanded as

$$I(\epsilon) = \int_{x_0}^{x_1} f(x, y + \epsilon\eta(x), y' + \epsilon\eta'(x))dx.$$

Assuming that the f function has continuous partial derivatives, the mean-value theorem is applicable:

$$f(x, y + \epsilon\eta(x), y' + \epsilon\eta'(x)) = f(x, y, y')+$$

$$\epsilon(\eta(x)\frac{\partial f(x,y,y')}{\partial y} + \eta'(x)\frac{\partial f(x,y,y')}{\partial y'}) + O(\epsilon^2).$$

By substituting we obtain

$$I(\epsilon) = \int_{x_0}^{x_1} f(x, y, y')dx+$$

$$\epsilon\int_{x_0}^{x_1} (\eta(x)\frac{\partial f(x,y,y')}{\partial y} + \eta'(x)\frac{\partial f(x,y,y')}{\partial y'})dx + O(\epsilon^2).$$

With the introduction of

$$\delta I_1 = \epsilon\int_{x_0}^{x_1} (\eta(x)\frac{\partial f(x,y,y')}{\partial y} + \eta'(x)\frac{\partial f(x,y,y')}{\partial y'})dx,$$

we can write

$$I(\epsilon) = I(0) + \delta I_1 + O(\epsilon^2),$$

where δI_1 is called the first variation. The vanishing of the first variation is a necessary, but not sufficient, condition to have an extremum. To establish a sufficient condition, assuming that the function is thrice continuously differentiable, we further expand as

$$I(\epsilon) = I(0) + \delta I_1 + \delta I_2 + O(\epsilon^3).$$