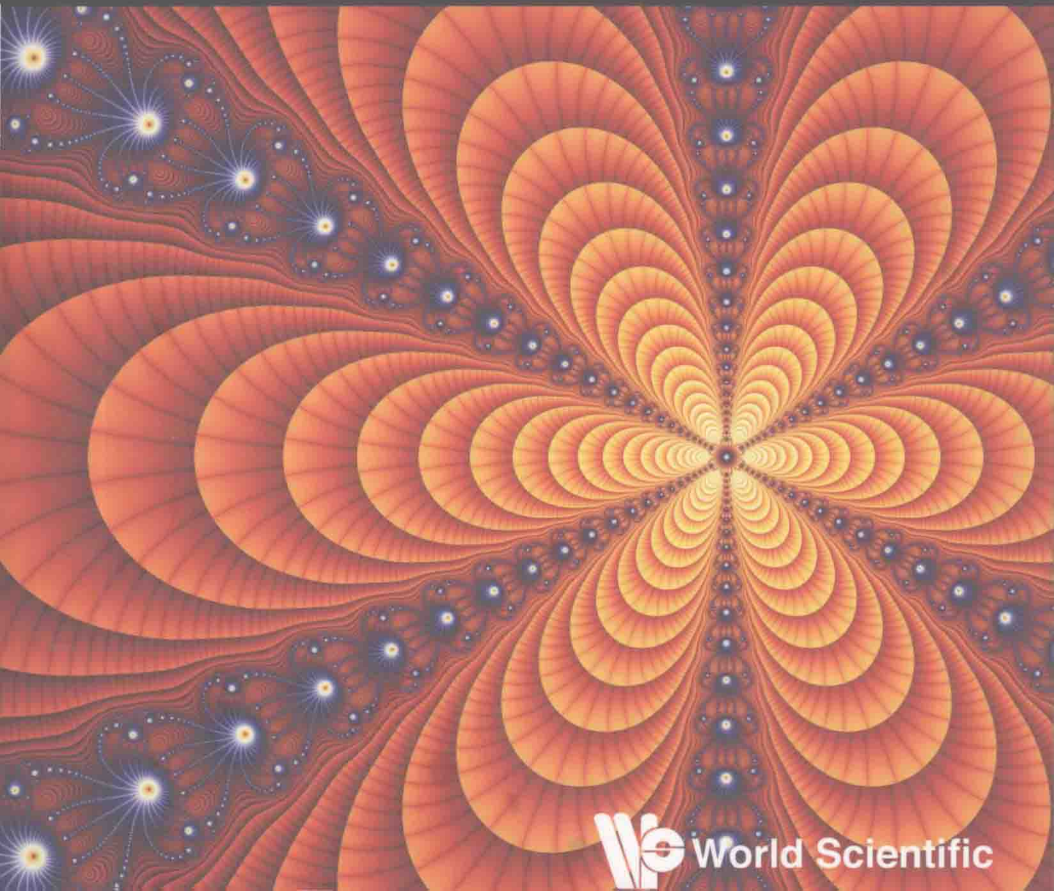
Functional Equations on Hypergroups

László Székelyhidi



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Published by

World Scientific Publishing Co. Pte. Ltd.

5 Toh Tuck Link, Singapore 596224

USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601

UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

FUNCTIONAL EQUATIONS ON HYPERGROUPS

ISBN 978-981-4407-00-7

Printed in Singapore by B & Jo Enterprise Pte Ltd

Preface

The theory of functional equations is one of the classical fields of mathematics. Functional equation problems arose in different areas from the ancient times both in theory and in applications. In 1966 J. Aczél published his book "Lectures on functional equations and their applications" (see [Acz66]), which is considered the bible of this theory. Although several books, monographs, papers, etc. have since then been published in the field, there is no doubt that this volume is still the most determining reference book. There are other important contributions by J. Aczél and J. Dhombres in "Functional Equations Containing Several Variables" (see [AD89]) and also a basic reference book is due to M. Kuczma ([Kuc09]). The interested reader will find several further references in these books on this wide-ranging field with applications in geometry, geometrical objects, statistics, information theory, utility theory, etc. Here we mention further volumes that have been published more recently, which may convince the reader of the usefulness and effectivity of the diverse methods and application possibilities of the theory of functional equations: [CRC92], [Cor02], [Cze02], [Fel08], [For10], [HIR98], [JS96], [Jár05], [Kan09], [SR98], [SK11], [Szé91].

In the old times functional equations were solved by different ad-hoc – however, ingenious – methods. Anyway, the theory was far from being a compact mathematical discipline in the sense that there were no real general solution methods, no real theories: a good idea would just solved the problem. Later on the situation changed. A pioneer work of A. Járai (see [Jár86], also in [JS96]) – in close connection with Hilbert's Fifth Problem – led to the observation that the strong algebraic character of a functional equation implies important consequences for the analytic behaviour of the solutions: namely, very weak analytic assumptions imply very strong analytic properties. This "regularization theory" was maybe the first important step to build up a coherent theory of functional equations together with its important consequences. The "good old ad-hoc" ideas were replaced by strong theorems and the weight of the theory of functional equations grew similar to that of the theory of differential equations and to other well-respected areas of mathematics. Beside several relevant works of Járai the interested reader will find further references in [JS96]. The comprehensive volume on regularization theory of Járai was published in 2005 [Jár05].

However, another stream started in the 90's with the monograph of the present author (see [Szé91]) emphasizing and introducing the fundamental role of spectral analysis and spectral synthesis in the theory of a special type of functional equations: the so-called convolution type functional equations. Convolution type functional equations are actually integral equations and it turns out that a major part of the so-called "classical" equations belongs to, or can be reduced to this type. In the monograph [Szé91] the author offers a general method for the solution of convolution type systems of functional equations. The essence of the method is that first the "basic building blocks" of the solution space of the functional equation should be found – these are the so-called "exponential monomials" – and then – in case of spectral synthesis – the linear combinations of these basic solutions will form a dense set in the solution space, that is, they characterize the solution space. It happens, or not, the exponential monomial solutions play a very special and important role in the solution process.

It turns out that several ideas of this type can be adopted to a more delicate situation: to the situation of hypergroups. The concept of DJS–hypergroup, which we shall use here (according to the initials of C. F. Dunkl, R. I. Jewett and R. Spector) is due to R. Lasser (see e.g. [Ros98], [BH95]), [Las83]. One can realize a hypergroup like the convolution structure of some measure algebra over a group, but the group structure has been neglected. If x, y are elements of a hypergroup, then the notation $x * y$ has a symbolic meaning only: it does not represent an element of the hypergroup, just a kind of "blurred product". In the group-case $x \cdot y$ is a well-defined element of the underlying structure, which also can be considered as a measure $\mu_{x,y}$ with the property that for any set B the value $\mu_{x,y}(B)$ is equal to 1 if $x \cdot y$ belongs to B and it is equal to 0, if $x \cdot y$ does not belong to B. Hence this is exactly the point mass concentrated at $x \cdot y$. However, in the case of a hypergroup, $x * y$ denotes a measure, actually $\delta_x * \delta_y$, which is not necessarily a point mass and $x * y(B)$ represents the "probability" of the event that the "product" $x * y$ belongs to B.

Anyway, using $x * y$ one can introduce translation operators on hypergroups, which makes it possible to set up a theory of harmonic analysis. The interested reader will find further details and references in the fundamental work of W. R. Bloom and H. Heyer ([BH95]). The theory of hypergroups has been a developing field, where ideas from different areas of mathematics can be utilized to obtain general results, which may help better understanding also in the classical situation. The interested reader will find further references and better insight in the papers [Las83], [Ros77], [Ros98], [AC11], [KPC10], [OEBG10], [Sam10], [Hey09], [LBPS09], [LBS09], [Mur08], [BH08], [DK07], [EL07], [Pav07c], [LOR07], [Mur07], [Mło06], [HK06], [BR05], [Las05], [FLS05], [SW03], [BBM02], [Gha02], [Las02], [BR02], [Kum01], [Hin00], [NI00], [GS00], [Ros99], [Pav99a], [Pav99b], [RV99], [CV99], [CS98], [Tri98], [Sch98], [RAL+98], [Par97], [Tri97], [OW96], [Wil95], [Rös95b], [Ros95c], [Hey95], [Ché95], [BK95], [CGS95].

As soon as translation operators appear on hypergroups a wide range of machinery can be adopted from the group-case. Nevertheless, the classical group-methods can be applied only restrictively: the special situation does not make it possible to "copy" the well-known classical methods. However, there are some distinguished function classes, like additive functions, exponential functions, or more generally, exponential polynomials, which play a vital role on both groups and hypergroups. Another class is represented by the so-called moment functions, which are extremely important in the different applications of hypergroups in probability theory and statistics. For more about these function classes the interested reader will find detailed information in [BH95], [Szé91], [Gal98], [Ros98], [Zeu92], [OS05], [OS04], [OS08], [Szé06c], [Gal97].

The appearance of translation operators enables us to utilize a very effective method of studying functional equations and systems of functional equations on hypergroups. Namely, it turns out that some of the methods of spectral analysis and spectral synthesis can be adopted and used in the hypergroup-situation. The present author has recently published a volume about the applications of spectral analysis and spectral synthesis on different structures ([Szé06a]). In that monograph the interested reader will find detailed information about spectral analysis, spectral synthesis and their use in the theory of functional equations. However, it turns out that several new ideas and methods can be transformed into the hypergroup-situation, which may enrich both fields: the theory of functional equations and the theory of hypergroups. To present the fruitful consequences of this delicate "marriage" was one of the main purpose to write this volume. The interested reader will find further results and references on these connections in [Ros98], [Gal98], [Las83], [Zeu92], [AC11], [RZ11], [Vaj10b], [KPC10], [OEBG10], [AK10], [HP10], [Hey09], [Las09a], [LBPS09], [NS09], [FK08], [Azi08], [Mur08], [BH08], [Ami07], [Pav07b], [HL07], [Pav07c], [Mur07], [Mło06], [HK06], [Men05], [BR05], [Las05], [FLS05], [Pav04], [SW03], [NI03], [HL03], [Wil02], [Gha02], [GT02b], [Gal02], [Las02], [GT02a], [Kum01], [NI01], [Hin00], [NI00], [GS00], [FL00], [Ros99], [Pav99a], [Pav99b], [NI99], [RV99], [CV99], [GS99], [Gal99], [Ren98], [NS98], [CS98], [Geb98], [Zeu98], [Tri98], [SW98], [Sch98], [Par97], [Pav97], [Wil97], [Zeu97], [KS97], [Flo96], [BH96], [Ren96], [BR96], [OW96], [Ehr96], [Sin96], [Zeu95b], [Wil95], [Rös95a], [CS95b], [Rös95b], [Her95], [HV95], [OEBB95], [Zeu95a], [Zeu95a], [Voi95], [Voi95], [CS95a], [BK95], [CGS95], [Szw95], [Han94], [RX94], [Zeu94], [Las94], [Voi93], [Hey93], [RX93], [LOR07], [BR02], [RAL+98].

In what follows we try to give a brief overlook about the structure of this booklet, the fundamental methods and the main results.

This Preface is followed by an Introduction in which we summarize the most important concepts concerning hypergroups. Some of these concepts are analogous to those of the ones in the group-case, but sometimes we meet basic differences. However, the concepts of additive functions, exponential functions, exponential monomials and exponential polynomials are introduced here and the relation to the corresponding group-case concepts is presented. Another important function class is the class of moment functions, mentioned above, which plays a very important role in the applications of hypergroups in probability theory and statistics. The interested reader should refer to [BH95], [Gal98], [Ros98], [Zeu92] and the references included in these works.

The Introduction is also devoted to present those analytic methods, which are very effective in the group case to prove strong regularity of solutions of functional equations assuming their weak regularity, only. The basic tools are Haar measure and invariant means. Here we tried to present a unified, nonstandard treatment of these basic analytic tools.

The next two chapters are devoted to the study of functional equations on a very important type of hypergroups: polynomial hypergroups in one variable and polynomial hypergroups in several variables. Here we give the complete description of those basic functional classes mentioned above, which play an important role in the applications: additive functions, exponential functions and exponential polynomials. Although the two chapters are very closely related to each other, the author's idea was to separate the consideration of polynomial hypergroups in one variable and in several variables. The reason is that sometimes the methods are basically different and this kind of "separation" may help better understanding. For detailed information about polynomial hypergroups the reader should refer to [Las83], [Szé04], [Vog87], [HHL10], [Las09b], [Las09a], [Szé08], [BH08], [Las07], [LOR07], [Mło06], [HL03], [BR02], [Tri00], [GS99], [Ehr96], [Zeu95b], [CS95a], [Szw95].

In Chapter 5 a new type of hypergroups appears: the so-called Sturm–Liouville hypergroups, which play a fundamental role in the theory of hypergroups, differential equations and initial value problems. It turns out that some of the above mentioned important function classes can be introduced, studied and characterized on these types of hypergroups. For more information about general and special Sturm–Liouville-hypergroups see e.g. [Ché95], [Szé06b], [OS08], [Vaj10b], [Vaj10a], [DK07], [Ole01], [Ché95], [Szé06b], [OS08], [Zeu89], [Ma08], [Tri05a], [Tri05b], [BBM02], [BX00b], [BX00a], [NRT98], [JT98], [BX98a], [BX98b], [BX97], [LT95], [BX95].

Chapter 6 contains three sections on the so-called two-point support hypergroups. Here we illustrate how the advanced methods of the theory of functional equations can be utilized to characterize some basic function classes on different types of hypergroups. We exhibit an example for two-point hypergroups of compact and of noncompact type, moreover another one, the so-called cosh *hypergroup*, which has been studied by H. Zeuner in [Zeu89].

Chapters 7 and 8 are – in some sense – the heart of this book: spectral analysis and spectral synthesis on different special types of hypergroups. Spectral analysis and spectral synthesis have become effective tools in functional equations recently. The classical roots go back to harmonic analysis and Fourier series. The abstract background can be found in [Loo53]. Basic knowledge and results on classical spectral theory of linear operators and spectral synthesis can be found in [Ben75], [Beu48], [Hel52], [Ris49], [Szé02], [Szé06a], [MA50], [Mal54], [Vog87], [DS88a], [DS88b], [DS88c], [Lef58], [Mal59], [Sch48], [Hel83], [HR63]. Studying harmonic analysis on hypergroups is possible because of the presence of translation operators. The fundamentals of this theory are presented in [BH95]. The use of harmonic analysis and synthesis in the theory of functional equations was invented in [Szé91]. However, the group-methods are not always easy to adopt in the hypergroup situation: sometimes new ideas are needed. Nevertheless, here the field is open as we were able to prove spectral analysis and spectral synthesis theorems for a restricted class of hypergroups, only. However, we hope that the applications we present here will convince the reader that further investigations in this area may lead to interesting and useful results – both for functional equationists and for hypergroup experts. We just mention that – as it is clear from the results of Sections 2.2, 3.2 and 4.2 – it is a nontrivial problem on how to define exponential monomials on arbitrary (commutative) hypergroups.

Chapter 9 is devoted to a classical problem of probability theory: the moment problem (see [Akh65], [Sti94]). We formulate the problem on commutative hypergroups and solve the uniqueness in the case of polynomial hypergroups in a single variable and of Sturm–Liouville-hypergroups.

In Chapters 10 and 11 we collected diverse applications of spectral analysis, spectral synthesis and other methods. These applications are illustrated on different classical and non-classical functional equations. For instance, in Chapter 11 the reader meets a new theory of difference equations on hypergroups – at least the basic and far-leading ideas.

The closing Chapter 12 is devoted to a special field of functional equations: stability theory. Since the pioneer talk of Stanislaw Ulam in 1940 presented to the audience of the Mathematics Club of the University of Wisconsin the door has been opened to a completely new world of investigations: stability became a central problem in the theory of functional equations (see [Cor02], [HIR98], [Hey93], [Cze02]). Here we make an attempt to outline some possible ways climbing these mountains on hypergroups.

This volume is completed with a list of references and a subject index.

We hope that the present work is able to represent faithfully the possibilities of connecting functional equation problems with those coming from the theory of hypergroups. We are convinced that both areas will profit from a "come together" of this type. This volume is written for those who have open eyes for both meadows, who have open ears for both concerts and who dare to enter a new world of ideas, a new world of methods – and, sometimes, a new world of unexpected difficulties.

The author is indebted to all those who helped in this work to become complete. Finally, I would like to express my special thanks to my students, Ágota Orosz and László Vajday, who did their best, who provided the newest results and without whose contribution this work could not have been accomplished.

László Székelyhidi
2012

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Chapter 1

Introduction

1.1 Basic concepts and facts

The major part of this section is taken from [BH95]. The concept of DJS–hypergroup (according to the initials of C. F. Dunkl, R. I. Jewett and R. Spector) depends on a set of axioms which can be formulated in several different ways. The way of formulating these axioms we follow here is due to R. Lasser (see e.g. [BH95], [Ros98]). One begins with a locally compact Hausdorff space K and with the space $\mathcal{C}_c(K)$ of all compactly supported complex valued functions on the space K. The space $\mathcal{C}_c(K)$ will be topologized as the *inductive limit* of the spaces

$$\mathcal{C}_E(K) = \{f \in \mathcal{C}_c(K) \; : \; \mathit{supp}\,(F) \subseteq E\},$$

where E is a compact subset of K carrying the uniform topology. A (complex) *Radon measure* μ is a *continuous linear functional* on $\mathcal{C}_c(K)$. Thus, for every compact subset E in K there exists a constant α_E such that $|\mu(f)| \leq \alpha_E ||f||_\infty$ for all f in $\mathcal{C}_E(K)$. The set of Radon measures on K will be denoted by $\mathcal{M}(K)$. For every μ in $\mathcal{M}(K)$ we write

$$||\mu|| = \sup\{|\mu(f)| \; : \; f \in \mathcal{C}_c(K), \, ||f||_\infty \leq 1\}.$$

A measure μ is said to be *bounded*, if $||\mu|| < +\infty$. In addition, μ is called a *probability measure*, if μ is nonnegative and $||\mu|| = 1$. The set of all bounded measures, the set of all compactly supported measures, the set of all probability measures and the set of all probability measures with compact support in $\mathcal{M}(K)$ will be denoted by $\mathcal{M}_b(K)$, $\mathcal{M}_c(K)$, $\mathcal{M}_1(K)$

and $\mathcal{M}_{1,c}(K)$, respectively. The point mass concentrated at x is denoted by δ_x. Via integration theory we are able to consider measures as functions on the σ-algebra $\mathcal{B}(K)$ of *Borel subsets* of K and we use the notation $\int_K f\,d\mu$ rather than $\mu(f)$ even when either is possible. We use the notation $\mathcal{M}_+(K)$ for the set of positive measures on the σ-algebra $\mathcal{B}(K)$ that means, for measures which take values in $[0, +\infty]$.

Now we formulate the first part of the axioms. Suppose that we have the following:

(H^*) There is a continuous mapping $(x, y) \mapsto \delta_x * \delta_y$ from $K \times K$ into $\mathcal{M}_{1,c}(K)$. This mapping is called *convolution.*

$(H^\vee)$ There is an involutive homeomorphism $x \mapsto x^\vee$ from K to K. This mapping is called *involution.*

(He) There is a fixed element e in K. This element is called *identity.*

Identifying x by δ_x the mapping in (H^*) has a unique extension to a continuous bilinear mapping from $\mathcal{M}_b(K) \times \mathcal{M}_b(K)$ to $\mathcal{M}_b(K)$. The involution on K extends to a continuous involution on $\mathcal{M}_b(K)$. Convolution maps $\mathcal{M}_1(K) \times \mathcal{M}_1(K)$ into $\mathcal{M}_1(K)$ and involution maps $\mathcal{M}_1(K)$ onto $\mathcal{M}_1(K)$. Then a *DJS–hypergroup*, or simply a *hypergroup* is a quadruple $(K, *, \vee, e)$ satisfying the following axioms: for each x, y, z in K we have

(H1) $\delta_x * (\delta_y * \delta_z) = (\delta_x * \delta_y) * \delta_z$,

(H2) $(\delta_x * \delta_y)^\vee = \delta_{y^\vee} * \delta_{x^\vee}$,

(H3) $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$,

(H4) e is in the support of $\delta_x * \delta_{y^\vee}$ if and only if $x = y$,

(H5) the mapping $(x, y) \mapsto supp\,(\delta_x * \delta_y)$ from $K \times K$ into the space of nonvoid compact subsets of K is continuous, the latter being endowed with the Michael topology (see [BH95]).

For any measures μ, ν in $\mathcal{M}_b(K)$ obviously $\mu * \nu$ denotes their convolution and $\mu^\vee$ denotes the involution of μ. With these operations $\mathcal{M}_b(K)$ is an *algebra with involution.* If the topology of K is discrete, then we call the hypergroup *discrete.* In case of discrete hypergroups the above axioms have a simpler form. As in this book we frequently will focus on discrete hypergroups, here we present a set of axioms for these types of hypergroups. Clearly, in the discrete case we can simply forget about the topological requirements in the previous axioms to get a purely algebraic system.

Let K be a set and suppose that the following properties are satisfied:

(D^*) There is a mapping $(x, y) \mapsto \delta_x * \delta_y$ from $K \times K$ into $\mathcal{M}_{1,c}(K)$, the space of all finitely supported probability measures on K. This mapping is called *convolution.*

($D^\vee$) There is an involutive bijection $x \mapsto x^\vee$ from K to K. This mapping is called *involution.*

(De) There is a fixed element e in K. This element is called *identity.*

Identifying x by δ_x as above and extending convolution and involution, a *discrete DJS–hypergroup* is a quadruple $(K, *, \vee, e)$ satisfying the following axioms: for each x, y, z in K we have

(D1) $\delta_x * (\delta_y * \delta_z) = (\delta_x * \delta_y) * \delta_z$,

(D2) $(\delta_x * \delta_y)^\vee = \delta_{y^\vee} * \delta_{x^\vee}$,

(D3) $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$,

(D4) e is in the support of $\delta_x * \delta_{y^\vee}$ if and only if $x = y$.

If $\delta_x * \delta_y = \delta_y * \delta_x$ holds for all x, y in K, then we call the hypergroup *commutative.* If $x^\vee = x$ holds for all x in K, then we call the hypergroup *Hermitian.* By (H2), any Hermitian hypergroup is commutative. In any case we have $e^\vee = e$. For instance, if $K = G$ is a locally compact Hausdorff–group, $\delta_x * \delta_y = \delta_{xy}$ for all x, y in K, $x^\vee$ is the inverse of x and e is the identity of G, then we obviously have a hypergroup $(K, *, \vee, e)$, which is commutative if and only if the group G is commutative. However, not every hypergroup originates in this way.

The simplest hypergroup is obviously the trivial one, consisting of a singleton. The next simplest hypergroup structure can be introduced on a set consisting of two elements. Now we describe all hypergroups of this type. Let $K = \{0, 1\}$. Clearly, the only Hausdorff topology on K is the discrete one. We specify $e = 0$ as the identity element. In this case the only involution satisfying the above axioms is the identity, that is, $0^\vee = 0$ and $1^\vee = 1$. Consequently, we have a Hermitian hypergroup, which is necessarily commutative. Now we have to define the four possible products $\delta_0 * \delta_0$, $\delta_0 * \delta_1$, $\delta_1 * \delta_0$ and $\delta_1 * \delta_1$. As δ_0 is the identity, the first three products are uniquely determined and the fourth one must have the form

$$\delta_1 * \delta_1 = \theta \cdot \delta_0 + (1 - \theta) \cdot \delta_1$$

with some number θ satisfying $0 \leq \theta \leq 1$. It turns out that $\theta \neq 0$, as a consequence of (D4). We shall denote this hypergroup by $D(\theta)$. It is clear that in this way we have a complete description of all possible hypergroup structures on a set consisting of two elements. Observe that in the case $\theta = 1$ we have a group isomorphic to $\mathbb{Z}_2$, the integers modulo 2, in any other case the resulting structure is not a group.

If K is any hypergroup and H is an arbitrary set, then for the function $f : K \to H$ we define $f^\vee$ by the formula

$$f^\vee(x) = f(x^\vee)$$

for each x in K. Obviously $\left(f^\vee\right)^\vee = f$. Any measure μ in $\mathcal{M}_b(K)$ satisfies

$$\mu^\vee(f) = \mu(f^\vee)$$

for any bounded Borel function $f : K \to \mathbb{C}$.

Let K be any hypergroup. Then, for each x, y in K the measure $\delta_x * \delta_y$ is a compactly supported probability measure on K, which makes the measurable space $(K, \mathcal{B}(K), \delta_x * \delta_y)$ a *probability space*. Any function $f : K \mapsto \mathbb{C}$, which is $\delta_x * \delta_y$-measurable, can be considered as a *random variable* on this probability space. In particular, any continuous complex valued function on K is a random variable with respect to any measure of the form $\delta_x * \delta_y$. Clearly, any f is integrable with respect to each δ_x and its *expectation* is

$$E_x(f) = \int f \, d\delta_x = f(x)\,,$$