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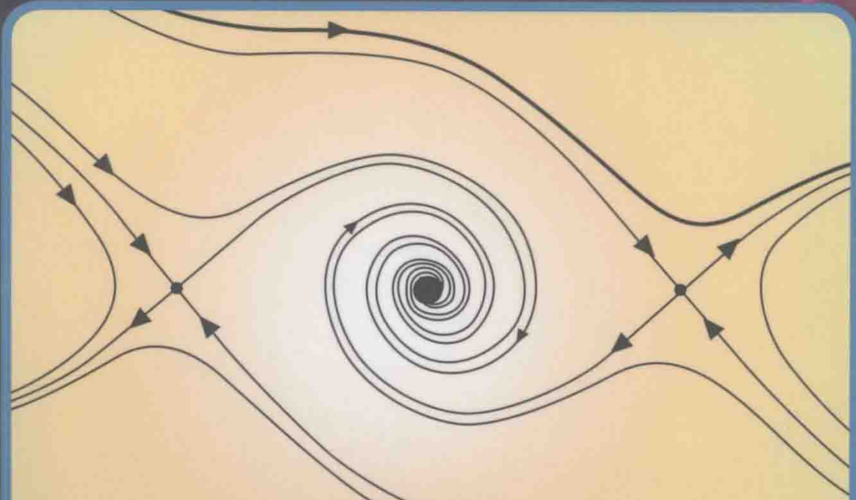
Lyapunov-Based Approach



Seyed Kamaleddin Yadavar Nikravesh

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Preface

The dynamic properties of a physical system can be described in terms of ordinary differential, partial differential, and difference equations, or any combination of these subjects. In addition, the systems can be time-varying, time-invariant and/or time-delayed, and continuous or discrete systems. These equations are often nonlinear in one way or the other and it is rarely possible to find their solutions. Numerical solutions for such nonlinear dynamic systems with an analog or digital computer are impractical. This is due to the fact that a complete solution must be carried out for every possible initial condition in the solution space. Graphical techniques, which can be employed for finding the solutions for special cases of first- and second-order ordinary systems, are not useful tools for other types of systems as well as higher-order ordinary systems. However, there are different theorems and methods concerning existence, uniqueness, stability, and other properties of nonlinear systems and/or their solutions. Among these qualitative properties, the stability of a given system is the most crucial systems issue. Without the guaranteed stability, the system will be of no value.

Many researchers have worked on stability robustness analysis for different systems. For a good list of these studies, one may read chapter five of sensitivity analysis by Eslami (e1). The aim of this book is to introduce some advanced tools for stability analysis of nonlinear systems. Toward this end, first, standard stability techniques are discussed with the shortcomings highlighted; then some recent developments in stability analysis are introduced, which can improve the applicability of standard techniques. Finally, stability analysis of special classes of nonlinear systems, for example, time-delayed systems and fuzzy systems, are proposed.

This book is organized as follows: In the first chapter, the stability of ordinary time-invariant differential equations will be considered. In Chapter 2, Lyapunov stability analysis will be studied. The subject of the third chapter is time-invariant systems. Chapter 4 deals with time-delayed systems. The stability analysis of fuzzy linguistic systems models is considered in Chapter 5.

This book is intended for graduate students of all disciplines who are involved in stability analysis of dynamic systems.

S.K.Y. Nikravesh

September 2010

(50th anniversary of the establishment of Amirkabir University of Technology [AUT])

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Contents

1 Basic Concepts

Introduction: In this chapter, the stability analysis of a system, the dynamics of which are represented in time domain by nonlinear time-invariant ordinary differential equations, is considered. This chapter consists of the following subsections:

1.1 Mathematical model for nonlinear systems.
1.2 Qualitative behavior of second-order linear time-invariant systems (LTI).

1.1 MATHEMATICAL MODEL FOR NONLINEAR SYSTEMS

A nonlinear system may mathematically be represented in the following form:

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m, t),\\
\dot{x}_2 &= f_2(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m, t),\\
\vdots\ \ & \ \ \vdots\\
\dot{x}_n &= f_n(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m, t),
\end{aligned} \tag{1.1}$$

where $\dot{x}_i,\ i = 1, 2, \ldots, n$ denotes the derivative of x_i (the ith state variable) with respect to the time variable t and u_j, $j = 1, 2, \ldots, m$ denote the input variables. Equation (1.1) could be written in the following state-space form:

$$\dot{x} = f(x, u, t), \tag{1.2}$$

where,

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} \quad \text{and} \quad f(x, u, t) = \begin{pmatrix} f_1(x, u, t) \\ f_2(x, u, t) \\ \vdots \\ f_n(x, u, t) \end{pmatrix}.$$

The measurable outputs (a p-dimensional vector) are functions of the states, the inputs, and the time such that:

$$\begin{aligned}
y_1 &= h_1(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m, t),\\
y_2 &= h_2(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m, t),\\
\vdots\ \ & \ \ \vdots\\
y_p &= h_p(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m, t).
\end{aligned} \tag{1.3}$$

or, in the following general form:

$$y = h(x,u,t). \tag{1.4}$$

Equations (1.2) and (1.4) together are called the *mathematical dynamic equations*, or:

$$\begin{aligned} \dot{x} &= f(x,u,t), \\ y &= h(x,u,t). \end{aligned} \tag{1.5}$$

These equations could be simulated using operational amplifiers (integrators) and function generators as shown in Figure 1.1 (a) and (b).

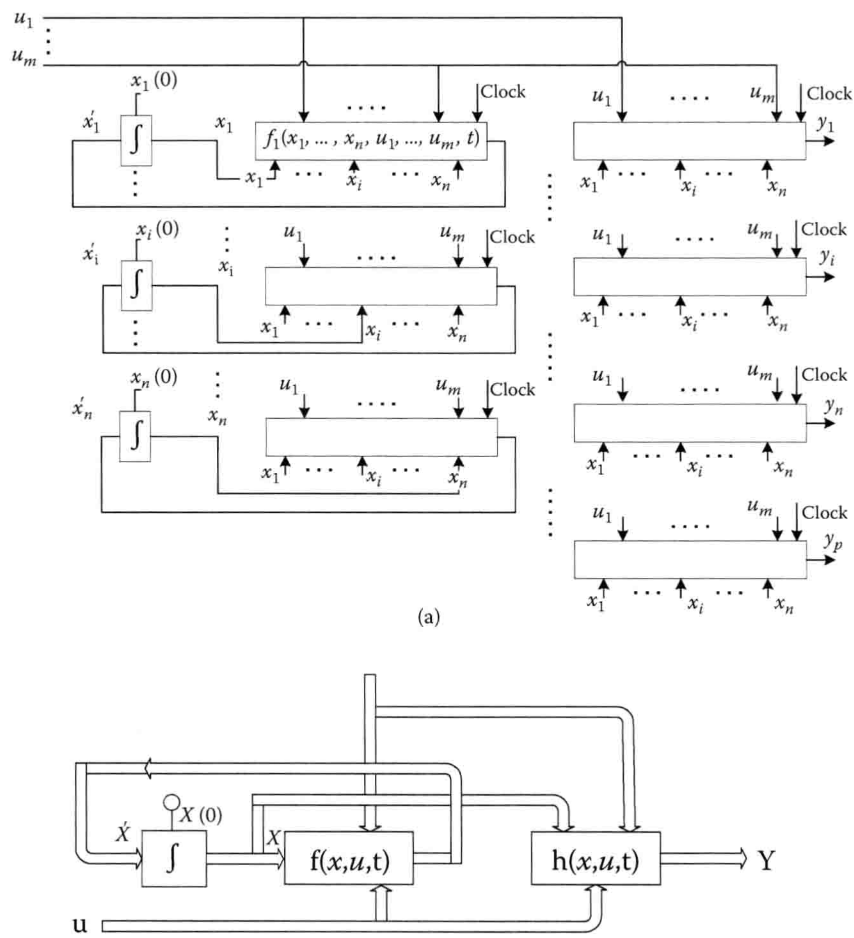

FIGURE 1.1 System dynamic simulation.

It seems the dynamic systems could be simulated to obtain their responses, having signal generators f_i and h_i. However, there is a drawback with this approach, since for each initial condition the simulation must be repeated. To have the actual response of dynamic systems, (1.2) and (1.4), the system must be at least locally Lipschitz in $x, \forall\, x \in D \subset R^n$ and continuous in t, for every t.

Throughout this book, wherever this type of dynamic equation occurs, the satisfaction of these conditions is assumed. The Lipschitz conditions are discussed shortly in this chapter.

Although in theory, the simulation could be proposed as a solution for the stability analysis, it is impractical or impossible, since in nonlinear system studies, every initial condition should be used.

Special Cases: If a system is a feedback system, then the system's inputs would be functions of the states, thus:

$$u \triangleq g(x,t). \tag{1.6}$$

Substituting (1.6) into (1.5) yields the following unforced dynamic equations:

$$\dot{x} = f(x,u,t) = F(x,t) \triangleq f(x,t), \;\; y = h(x,u,t) = H(x,t) \triangleq h(x,t), \tag{1.7a}$$

If the dynamic system (1.7a) is time invariant, then the system is called an *autonomous* (either *forced* or *unforced*) system.

$$\begin{aligned} \dot{x} &= f(x,u), \quad \text{or} \quad f(x) \\ y &= h(x,u), \quad \text{or} \quad h(x). \end{aligned} \tag{1.7b}$$

If the linearization technique is used in dynamic equations (1.5) or (1.7b), then linear time-varying (1.8) or linear time-invariant (forced or unforced) (1.9) equations yield:

$$\begin{aligned} \dot{x}_n &= \left(\frac{\partial f}{\partial x}\Big|_{\substack{x0\\u0}}\right)x_n + \left(\frac{\partial f}{\partial u}\Big|_{\substack{x0\\u0}}\right)u_n \triangleq A(t)\,x_n + B(t)\,u_n, \\ y_n &= \left(\frac{\partial h}{\partial x}\Big|_{\substack{x0\\u0}}\right)x_n + \left(\frac{\partial h}{\partial u}\Big|_{\substack{x0\\u0}}\right)u_n \triangleq C(t)x_n + D(t)u_n, \end{aligned} \tag{1.8}$$

or:

$$\begin{aligned} \dot{x}_n &\triangleq Ax_n + Bu_n, \\ y_n &\triangleq Cx_n + Du_n. \end{aligned} \tag{1.9}$$

The index "n" stands for new variable. Note that (1.8) or (1.9) can only predict the local behavior of the nonlinear system of (1.5) or (1.7), respectively.

1.1.1 Existence and Uniqueness of Solutions [k1]

The existence and uniqueness of the solution of (1.6) are given by the following theorem.

Theorem 1.1:

Let $f(x,t)$ be a single valued continuous function in a region defined by $|x_i - x_i(o)| < h_i,\ i = 1,2,\ldots,n$ and $o \le t - t_1 < T$ in which $|f(x,t)| < M$ for some $o<M<\infty$, and t_1 is the domain of piecewise continuity of $f(x,t)$. If $f(x,t)$ satisfies the following Lipschitz condition in x:

$$\|f(x_1,t) - f(x_2,t)\| \le L\|x_1 - x_2\| \quad ,o < L < \infty,$$

$$\forall x_1, x_2 \in B = \{x \in R^n \ \|x - x_o\| \le r\}, \forall t \in (t_o, t_1), \quad r > o,$$

then there exists some $\delta > o$ such that the state equation $\dot{x} = f(x,t)$ with $x(t_o) = x_o$ has a unique solution over $[t_o, t_o + \delta]$; $\delta = \min(T, \frac{h_i}{M})$. ■

When n = 1 and $f(x)$ is autonomous, then the Lipschitz condition implies,

$$\frac{|f(x_1) - f(x_2)|}{|x_1 - x_2|} \le L,$$

that is, in a plane of $f(x)$ versus x, a straight line joining any two points of $f(x)$ cannot have a slope with absolute value greater than L. Therefore, a discontinuous function is not locally Lipschitz at the points of discontinuity.

More generally, if for $t \in I \subset R$ and $x \in D \subset R^n$, $f(x,t)$ and its partial derivatives $\partial f_i / \partial x_j$ are continuous, then $f(x,t)$ is locally Lipschitz in x on D. $f(x,t)$ is globally Lipschitz in x if and only if (iff) $\frac{\partial f_i}{\partial x_j}$ are globally uniformly bounded in t.

Example 1.1:

Note that $\dot{x} = f(x) = x^{1/3}$ is not locally Lipschitz, at $x = o$ since:

$$f'(x) = \frac{1}{3}x^{-2/3} \to \infty \quad as\ x \to 0 \quad x(t) = \left(\frac{2t}{3}\right)^{\frac{3}{2}} \quad \text{and } x(t) = 0$$

are the two different solutions for this differential equation, when the initial state is

$$x(0) = 0.$$

■

Also, $\dot{x} = f(x) = -x^2$ is locally Lipschitz for all x but not globally Lipschitz, because $f'(x) = -2x$ is not globally bounded.

Note that the linear time-varying system:

$$\dot{x} = A(t)x + b(t)u,$$

is globally Lipschitz if and only if (iff) the elements of *A(t)* are piecewise continuous and bounded. Therefore, the linear time-invariant systems are all globally Lipschitz.

In the following, it is assumed that the systems under consideration satisfy the Lipschitz conditions. If the equilibrium state is at $x_e \neq 0$, then let

$$y \triangleq x - x_e,$$

thus:

$$\dot{y} = \dot{x} = f(x) = f(y + x_e) \triangleq f_e(y),$$

where $f_e(0) = 0$. Therefore, without loss of the generality, the origin could be considered as an isolated equilibrium state.

Equilibrium States: These are the states "x_e" that if an unforced system (with neither control inputs nor disturbance) reaches every one of these states, it will stay there forever; therefore,

$$\dot{x}_e = f(x_e, u = 0, t) = 0, \qquad \forall t. \tag{1.10}$$

In a linear system with nonsingular *A(t)*, the sole equilibrium state is the origin. In the nonlinear case, the equilibrium state could be an isolated one, similar to a linear system, or infinitely many isolated equilibrium states, or there could be a continuum of equilibrium states.

1.2 QUALITATIVE BEHAVIOR OF SECOND-ORDER LINEAR TIME-INVARIANT SYSTEMS

Consider the unforced system:

$$\dot{x} = Ax. \tag{1.11}$$

The eigenvalues of *A* may satisfy one of the following situations:

a_1. Both real negative with $\lambda_2 < \lambda_1 < 0$.
a_2. Both real positive with $\lambda_2 > \lambda_1 > 0$.

b. Real eigenvalues with opposite signs, that is, $\lambda_2 < 0 < \lambda_1$.
c. Complex eigenvalues $\lambda_{1,2} = \alpha \pm j\omega$.

The typical family of trajectories of these situations is shown accordingly in Figure 1.2. The proof is omitted here and interested readers are referred to the literature for classical nonlinear control systems.

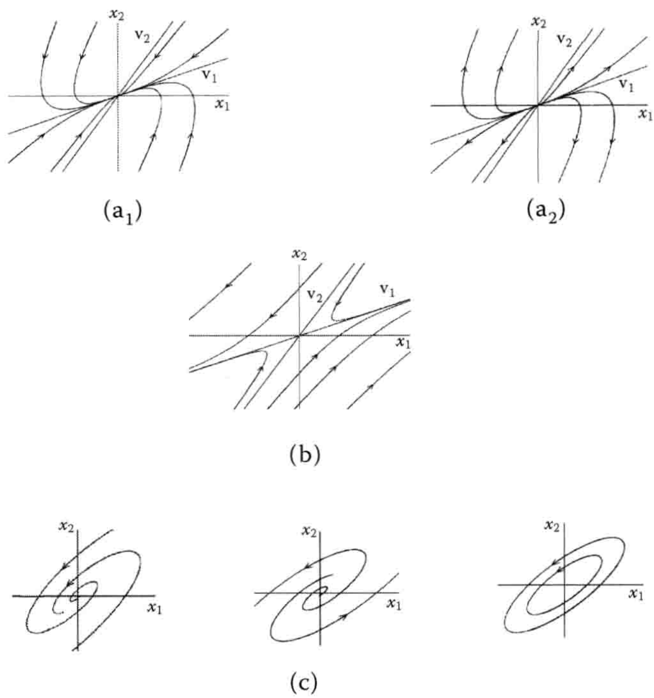

FIGURE 1.2 Typical trajectories of second-order dynamic systems.

Note that when $\lambda_{1,2} = \pm j\omega$ (i.e., $Re\ [\lambda_i] = o$ for some i) the linearization technique does not work [k1].

Example 1.2 [k1]:

Consider the following inverted pendulum equation with friction:

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = -a\sin x_1 - bx_2,$$

The equilibrium states are as follows:

$$x_1 = k\pi, \qquad k = 0,1,\dots,$$

$$x_2 = 0.$$

The unforced linearized system would be as follows:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a\cos x_1 & -b \end{pmatrix}_{0,0} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a & -b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

therefore:

$$\det(\lambda 1 - A) = \det\begin{pmatrix} \lambda & -1 \\ a & \lambda + b \end{pmatrix} = \lambda^2 + b\lambda + a = 0 \quad \Rightarrow \lambda = -\frac{b}{2} \pm \frac{1}{2}\sqrt{b^2 - 4a},$$

thus, for both a and b positive, the eigenvalues have negative real parts. Therefore, the origin is asymptotically stable (node).

To determine the stability of the equilibrium state at $(\pi,0)$, the Jacobian matrix would be evaluated at that state.

$$A = \begin{bmatrix} 0 & 1 \\ a & -b \end{bmatrix} \Rightarrow \lambda^2 + b\lambda - a = 0,$$

$$\lambda_{1,2} = -\frac{1}{2}b \pm \frac{1}{2}\sqrt{b^2 + 4a},$$

For positive scalars a and b, one of the eigenvalues is in the open right-half plane, which implies unstable equilibrium state. Figure 1.3 represents the phase plane, separatrices, and equilibrium state (stable nodes and saddle points).

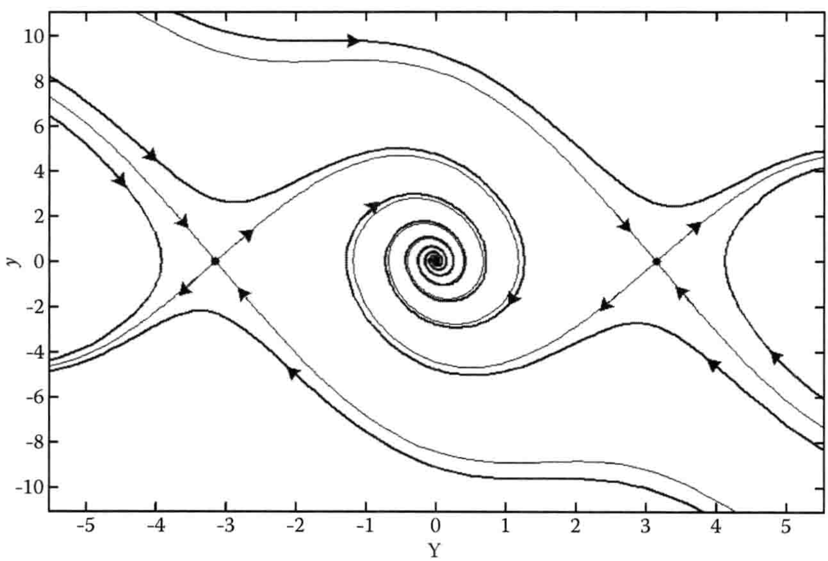

FIGURE 1.3 Phase plane of Example 1.2.

PROBLEMS:

1.1: Consider the following scalar system:

$$\dot{x} = -(1+x^2)\,x^3 \triangleq f(x)$$

a. Is $f(x)$ locally Lipschitz?
b. Is $f(x)$ globally Lipschitz? ■

1.2: Consider the following vector-valued system:

$$f(x) = \begin{bmatrix} x_2 \\ -sat(x_1 + x_2) \end{bmatrix}$$

where:

$$sat(x) = \begin{cases} -1 & x < -1 \\ x & |x| \le 1 \\ +1 & x > 1 \end{cases}$$

Does $f(x)$ satisfy the Lipschitz condition? ■

1.3: The nonlinear dynamic equation for a pendulum is given by:

$$ml\ddot{\theta} = -mg\sin\theta - kl\dot{\theta},$$

where l is the length of the pendulum, m is the mass of the ball, and θ is the angle suspended by the rod and the vertical axis through the pivot point.

a. Choose appropriate state variables and write down the state equation.
b. Find all equilibrium states of the system.
c. Linearize the system around the equilibrium states, and determine whether the system equilibrium states are stable or not.
d. Rewrite the pendulum model into the feedback connection form.
e. Make a simulation model of the system in Simulink®. Simulate the system from various initial states. Is the equilibrium state of the system stable? Are the equilibrium states unique? Explain the physical intuition behind your findings.
f. Use the function Linmod in MATLAB® to find the linearized models for the equilibrium states. Compare with the linearization that you derived. ■

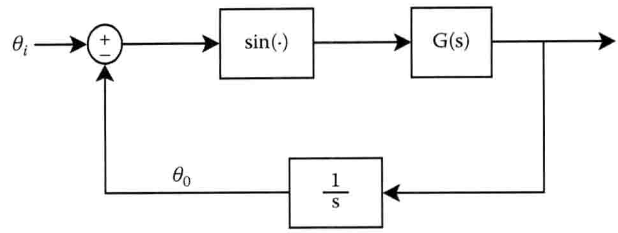

FIGURE P1.4 The phase-locked loop.

1.4: A phase-locked loop system can be represented by the block diagram of Figure P1.4. Let $\{A,B,C\}$ be a state-space representation of the transfer function $G(s)$. Assume that all eigenvalues of A have negative real parts, $G(0)\neq 0$ and θ_i is constant. Let z be the state of the realization $\{A,B,C\}$.

a. Show that:

$$\dot{z} = Az + B\sin e$$

$$\dot{e} = -Cz$$

is a state equation for the closed-loop system.

b. Find all equilibrium states of the system.
c. Show that if $G(s) = 1/(\tau s+1)$, the closed-loop model coincides with the model of a pendulum equation. ■

1.5: A synchronous generator connected to an infinite bus can be modeled by:

$$M\ddot{\delta} = P - D\dot{\delta} - \eta E_q \sin\delta$$

$$\tau\ddot{E}_q = -\eta_2 E_q + \eta_3 \cos\delta + E_{fd},$$

where δ is the angle in radians, E_q is voltage, P is mechanical input power, E_{fd} is field voltage (input), D is damping coefficient, M is inertial coefficient, τ is the time constant, and η_1, η_2 , and η_3 are constant parameters.

a. Using δ, $\dot{\delta}$, and E_q as state variables, find the state equation.
b. Suppose that τ is relatively large so that $\dot{E}_q \approx 0$. Show that assuming E_q to be constant reduces the model to a pendulum equation.
c. For the simplified model, derived in Problem 1.5(b), find all equilibrium points. ■

1.6: A mass-spring system is shown in Figure P1.6. The displacement, y, from a reference point is given by :

$$m\ddot{y} + F_f + F_{sp} = F$$