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Jürgen Leiterer

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PREFACE

The contemporary analysis on complex manifolds has been developed by means of the theory of coherent analytic sheaves (see, for instance, [Gunning/Rossi 1965 and Grauert/Remmert 1971,1977,1984]) and/or harmonic analysis (see, for instance, [Chern 1956, Kohn 1964, Hörmander 1966, Morrey 1966, Wells 1973]). However, the first fundamental contributions to the function theory of several complex variables, obtained in the period 1936-51 by K.Oka, were based on the classical constructive method of integral representations (see [Oka 1984]).

In the seventies this constructive approach has had a come-back in order to obtain in a strengthened form (with uniform estimates) the main results of the theory of functions on complex manifolds. A systematic development of the function theory in C^n which uses integral formulas as the principal tool was given in the books [Henkin/Leiterer 1984 and Range 1986].

The constructive method of integral representations is working with success also in several other fields of complex analysis: Cauchy-Riemann cohomology of complex manifolds, holomorphic vector bundles on complex manifolds, analysis on Cauchy-Riemann manifolds, Radon-Penrose transform, inverse scattering problem.

The authors intend to write a book where these fields will be presented from the viewpoint of integral formulas. The present monograph is a tentative version of the first part of that book. Here we develop in detail the basic facts on the Cauchy-Riemann cohomology of complex manifolds, where the emphasis is on finiteness, vanishing, and separation theorems for a class of complex manifolds which lies between the Stein, and the compact manifolds. Theorems A and B of Oka-Cartan for Stein manifolds as well as the finiteness theorems of Kodaira for compact, and Grauert for pseudoconvex manifolds appear as special cases of more general theorems.

The theory developed in the present monograph was mainly obtained in the articles [Andreotti/Grauert 1962, Andreotti/Vesentini 1965, Andreotti/Norguet 1966, Kohn/Rossi 1965, and Hörmander 1965] (it is astonishing that these remarkable results did not as yet enter into books). The novelty added here consists in new proofs based on integral formulas. As in the case of the theory of functions in $\mathbb{C}^n$, this makes it possible to prove all basic facts in a strengthened form: uniform estimates for solutions of the Cauchy-Riemann equation for differential forms on strictly q-convex and strictly q-concave domains, uniform approximation and uniform interpolation for the $\bar{\partial}$-cohomology classes on strictly q-convex domains, solution of the E.Levi problem for the $\bar{\partial}$-cohomology with uniform estimates, the Andreotti-Vesentini separation theorem with uniform estimates etc. A part of these results with uniform estimates was obtained already in the seventies [Fischer/Lieb 1974, Ovrelid 1976, Henkin 1977, Lieb 1979]. Some of these reusults are new.

These results with uniform estimates admit important applications in the theory of holomorphic vector bundles and the theory of the tangent Cauchy-Riemann equation. Such applications will be the subject of the following parts of the pending book mentioned above - elements of this are contained in the articles [Ajrapetjan/Henkin 1984 and Henkin/Leiterer 1986]. Moreover, in the following parts of that book, we intend to present some developments of the Andreotti-Grauert theory in connection with the Radon-Penrose transform - elements of this can be found in [Henkin 1983, Henkin/Poljakov 1986 and Leiterer 1986].

Note. Further on the book [Henkin/Leiterer 1984] will be refered to as [H/L]. The present monograph may be considered as a continuation of [H/L]. However, without proof, we use only results from the elementary Chapter 1 of [H/L]. Moreover, all basic results of Chapter 2 of [H/L], which is devoted to the theory of functions on completely pseudoconvex manifolds (= Stein manifolds, after solution of the E.Levi problem), are obtained anew, as the special case q=n-1 of Chapter 3 of the present work.

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CONTENTS

CHAPTER I. INTEGRAL FORMULAS AND FIRST APPLICATIONS

Summary. This chapter is devoted to the general theory of integral representation formulas for solutions of the Cauchy-Riemann equation in $\mathbb{C}^n$. In Sect. 0 we introduce notations and standard definitions. In Sect. 1, first we recall basic facts about the Martinelli-Bochner-Koppelman formula, where, for some of the proofs we refer to Chapter 1 in [H/L]. Then, by means of this formula, we prove the regularity of the $\bar\partial$-operator as well as the Kodaira finiteness theorem on compact complex manifolds. Sect. 2 is devoted to the Cauchy-Fantappie formula - a generalization of the Martinelli-Bochner-Koppelman formula. (For a direct proof of this formula, again we refer to Chapter 1 in [H/L]. Notice also that in Sect. 3 of the present work the full proof of a more general formula will be given.) Then, as an application of the Cauchy-Fantappie formula, we prove the Poincaré $\bar\partial$-lemma. At the end of Sect. 2, we recall the arguments which lead from the Poincaré $\bar\partial$-lemma and the regularity of the $\bar\partial$-operator to the Dolbeault isomorphism and the theorem on smoothing of the $\bar\partial$-cohomology. In Sect. 3 we prove a generalization of the Cauchy--Fantappie formula, which will be called the piecewise Cauchy-Fantappie formula. This formula is especially useful for domains with a boundary which consists of several pieces each of which has its own "advantages" as, for instance, q-convexity, q-concavity, Levi flatness. In Chapters III and IV, appropriate special cases of this formula will be used to solve the Cauchy-Riemann equation with uniform estimates.

0. Generalities about differential forms and currents

0.1. Some notations. $\mathbb{C}$ is the complex plane, and $\mathbb{C}^n$ is the n-dimensional complex Euclidean space. If $x \in \mathbb{C}^n$, then by $x_1, \ldots, x_n$ we denote the canonical complex coordinates of x. We write

$$\langle x, y\rangle = x_1 y_1 + \ldots + x_n y_n \quad \text{and} \quad |x| = (|x_1|^2 + \ldots + |x_n|^2)^{1/2}$$

for $x,y\in\mathbb{C}^n$. $\mathbb{R}$ is the real line, and $\mathbb{R}^n$ is the n-dimensional real Euclidean space.

The word "domain" means an arbitrary (not necessarily connected) open set. We write $A \subset\subset B$ to say that A is a relatively compact subset of B. The word "neighborhood" always means an open neighborhood.

The notion of a differential form (or a form) will be used for differential forms with measurable complex-valued coefficients. The degree of a differential form f will be denoted by deg f, and its support by supp f. Continuous forms will be called also C^0 forms. C^k forms $(k=1,2,\ldots,\infty)$ are forms with k times continuously differentiable coefficients.

A form f defined on a complex manifold is called an (r,s)-form (or a form of bidegree (r,s)) if, with respect to local holomorphic coordinates $z_1,\ldots,z_n$,

$$f = \sum_{|J|=r,\,|K|=s}{}' f_{JK}\, dz^J \wedge d\bar{z}^K. \tag{0.1}$$

Here the summation is over all strictly increasing r-tuples $J=(j_1,\ldots,j_r)$ and all strictly increasing s-tuples $K=(k_1,\ldots,k_s)$ in $\{1,\ldots,n\}$, $dz^J = dz_{j_1}\wedge\ldots\wedge dz_{j_r}$, $d\bar{z}^K = d\bar{z}_{k_1}\wedge\ldots\wedge d\bar{z}_{k_s}$, and the coefficients f_{JK} are complex-valued functions.

As usual, by d we denote the exterior differential operator. On complex manifolds, by $\bar\partial$ we denote the Cauchy-Riemann operator, and we set $\partial := d-\bar\partial$, i.e. if f is as in (0.1), then

$$\bar\partial f := \sum_{|J|=r,\,|K|=s}{}' \sum_{l=1}^{n} \frac{\partial f_{JK}}{\partial \bar{z}_l}\, d\bar{z}_l \wedge dz^J \wedge d\bar{z}^K$$

and

$$\partial f := \sum_{|J|=r,\,|K|=s}{}' \sum_{l=1}^{n} \frac{\partial f_{JK}}{\partial z_l}\, dz_l \wedge dz^J \wedge d\bar{z}^K,$$

where

$$\frac{\partial}{\partial \bar{z}_l} := \frac{1}{2}\left(\frac{\partial}{\partial t_l} + i\frac{\partial}{\partial t_{l+n}}\right) \quad\text{and}\quad \frac{\partial}{\partial z_l} := \frac{1}{2}\left(\frac{\partial}{\partial t_l} - i\frac{\partial}{\partial t_{l+n}}\right)$$

with $t_l :=$ real part of z_l, and $t_{l+n} :=$ imaginary part of z_l.

Remark. The main subject considered in this monograph are differential forms with values in holomorphic vector bundles (for the notion of forms with values in vector bundles, see, for instance [Wells 1973]). Since any (s,r)-form with values in a holomorphic vector bundle E may be identified with a (0,r)-form with values in the holomorphic vector bundle $\Lambda^{s,0}\otimes E$, where $\Lambda^{s,0}$ is the sth exterior power of the holomorphic cotangent bundle, in the most cases, we shall restrict ourselves to (0,r)-forms (but with coefficients in arbitrary holomorphic vector bundles).

0.2. Integration with respect to a part of the variables. Suppose X, Y are real C^1 manifolds, and f is a differential form on $X\times Y$. Let $m = \dim_{\mathbb{R}} X$, $n = \dim_{\mathbb{R}} Y$, and let $y_1,\ldots,y_n$ be local C^1 coordinates in some open $U \subseteq Y$. Consider the unique representation

$$f(x,y) = \sum_I{}' f_I(x,y)\wedge dy^I, \qquad x\in X,\ y\in U,$$

where the summation is over all strictly increasing r-tuples $I=(i_1,\ldots,i_r)$ in $\{1,..,n\}$ with $r \leq \deg f$, $dy^I := dy_{i_1}\wedge\ldots\wedge dy_{i_r}$, and $f_I(x,y)$ is a differential form of degree $\deg f - r$ on X which depends on $y\in U$. If X is oriented and the integrals $\int_X f_I(x,y)$ exist for all fixed $y\in U$ and any strictly increasing r-tuple I in $\{1,\ldots,n\}$ with $r = \deg f - m$, then we define

$$\int_X f(x,y) = \sum_I{}' \left(\int_{x\in X} f_I(x,y)\right) dy^I, \qquad y\in U,$$

where the summation is over all strictly increasing r-tuples in $\{1,\ldots,n\}$ with $r = \deg f - m$. The result of this integration is a differential form of degree $\deg f - m$ on U.

This form is independent of the choice of the local coordinates $y_1,\ldots,y_n$. Therefore $\int_X f(x,y)$ is well-defined for all $y\in Y$. Notice that, by this definition, $\int_X f(x,y) = 0$ if f does not contain monomials which are of degree m in x.

0.3. The absolute value |f| of a differential form f of maximal degree. Let X be an oriented real C^1 manifold of real dimension m, and let f be a differential form of degree m on X. Then by $|f|$ we denote the differential form on X which is defined as follows: If $x_1,\ldots,x_m$ are positively oriented C^1 coordinates in some open $U \subseteq X$ and F is the complex-valued function on U with $f = F dx_1\wedge\ldots\wedge dx_m$, then

$$|f| := |F| dx_1\wedge\ldots\wedge dx_m \quad \text{on } U.$$

We remark that if f is integrable, then

$$\left| \int_X f \right| \leq \int_X |f|.$$

If g is a second differential form of degree m on X, then we write

$$|g| \leq |f| \quad \text{on } X$$

if the following condition is fulfilled: If $x_1, \ldots, x_m$ are positively oriented C^1 coordinates in some open $U \subseteq X$, and G, F are the functions on U with $g = G dx_1 \wedge \ldots \wedge dx_m$ and $f = F dx_1 \wedge \ldots \wedge dx_m$, then $|G| \leq |F|$ on U.

0.4. The Riemannian norm of a differential form at a point. Let f be a differential form of degree m defined on a domain $D \subseteq \mathbb{R}^n$. If m=0, i.e. f is a complex valued function, then we write $\|f(z)\| = |f(z)|$, $z \in D$. If m>0, $x_1, \ldots, x_n$ are the canonical coordinates in $\mathbb{R}^n$, and

$$f = \sum_{1 \leq i_1 < \ldots < i_m \leq n} f_{i_1, \ldots, i_m} dx_{i_1} \wedge \ldots \wedge dx_{i_m},$$

then we define

$$\|f(z)\| = \left(\sum_{1 \leq i_1 < \ldots < i_m \leq n} \left| f_{i_1, \ldots, i_m}(z) \right|^2 \right)^{1/2}, \qquad z \in D.$$

Now let f be an arbitrary real C^1 manifold. Then we choose a locally finite open covering $\{U_j\}$ of X together with C^1 coordinates $x_1^j, \ldots, x_n^j$ on U_j. Further, we choose a continuous partition of unity $\{\chi_j\}$ subordinated to $\{U_j\}$. If f is a differential form in a neighborhood of some point $z \in X$ and

$$f(z) = \sum_{1 \leq i_1 < \ldots < i_m \leq n} f^j_{i_1, \ldots, i_m}(z) dx^j_{i_1}(z) \wedge \ldots \wedge dx^j_{i_m}(z)$$

for all j with $z \in U_j$, then we define

$$\|f(z)\| = \sum_j \chi_j(z) \left[\sum_{1 \leq i_1 < \ldots < i_m \leq n} \left| f^j_{i_1, \ldots, i_m}(z) \right|^2 \right]^{1/2}.$$

Of course, this definition depends on the choice of the local coordinates as well as on the choice of the partition of unity. Any norm obtained in this way will be called a Riemannian norm on X (Riemannian metric). If $\|\cdot\|_1$ and $\|\cdot\|_2$ are two Riemannian norms on a C^1 manifold X, then, clearly, for any compact set $K \subset\subset X$, one can find constants $c>0$ and $C<\infty$ such that

$$c\|f(x)\|_1 \leq \|f(x)\|_2 \leq C\|f(x)\|_1 \qquad (0.2)$$

for all $x\in K$ and all differential forms f on X. In the present monograph, in all cases where we meet Riemannian norms, estimate (0.2) ensures independence of the special choice of this norm. Therefore we shall use the following

Convention. The cotangent bundle of any real C^1 manifold is assumed to be endowed with an (arbitrary but fixed) Riemannian norm. □

Finally, we want to generalize the notion of a Riemannian norm to forms with values in vector bundles. Let E be a vector bundle of rank N over a real C^1 manifold X. Then a Riemannian norm in E is given by a locally finite open covering $\{U_j\}$ of X together with a family of vector bundle trivializations

$$E|U_j = U_j\times\mathbb{C}^N \qquad (0.3)$$

as well as a C^1 partition of unity $\{\chi_j\}$ subordinated to $\{U_j\}$. If f is an E-valued differential form (i.e. a measurable section of $\Lambda^m\otimes E$, where m is the degree of f and Λ^m the mth exterior power of the cotangent bundle of X) in a neighborhood U_z of some point $z\in X$ and $(f_1^j,\dots,f_N^j)$ is the vector of differential forms on $U_z\cap U_j$ which represents $f|U_z\cap U_j$ via (0.3), then we define the Riemannian norm $\|f(z)\|$ of $f(z)$ by

$$\|f(z)\| = \sum_j \chi_j(z)\left(\sum_{k=1}^{N}\|f_k^j(z)\|^2\right)^{1/2}, \quad z\in X. \qquad (0.4)$$

(The norms $\|f_k^j(z)\|$ are well-defined, since, by our Convention, X is already endowed with a Riemannian norm.) The same arguments as above now lead to the following convention (which will be used throughout this monograph):

Convention. If E is a vector bundle over a real C^1 manifold X, then we assume that E is endowed with an (arbitrary but fixed) Riemannian norm.

0.5. Determinants of matrices of differential forms. Let $A = (a_{ij})_{i,j=1}^n$ be a quadratic matrix whose elements a_{ij} are differential forms. Then we define a differential form det A by setting

$$\det A = \sum_{\sigma} \operatorname{sgn}(\sigma)a_{\sigma(1),1}\wedge\dots\wedge a_{\sigma(n),n},$$

where the summation is over all permutations σ of $\{1,\dots,n\}$ and $\operatorname{sgn}(\sigma)$

is the signature of σ. Under this definition the usual relations between row operations (the first index is the row index) and determinants hold true. Therefore, as in the case of usual determinants one obtains that if A is an n×n matrix of differential forms and Z is an n×n matrix of complex-valued functions, then det ZA = det Z det A. Notice that, in general, det AZ ≠ det A det Z. Further, it is possible that det A = 0 although some of the columns of A are equal. For references, in the following proposition we collect some useful facts about such determinants (the proofs are obvious):

<u>0.6. Proposition.</u> (i) The determinant of a quadratic matrix of differential forms is a multi-linear map of the columns with respect to linear combinations whose coefficients are complex-valued functions.

(ii) If A is an n×n-matrix of differential forms and if, for some $1 \le k < l \le n$, $a_{ik}=z_i b_k$ and $a_{il}=z_i b_l$ for all i, where b_k, b_l are arbitrary differential forms and z_i are complex-valued functions, then det A = 0.

(iii) Let a_i and a_j be two columns in a quadratic matrix A of differential forms such that all forms in a_i have the same degree d_i and all forms in a_j are of the same degree d_j. Further, let $\tilde{A}$ be the matrix obtained from A by interchanging these two columns. Then

$$\det \tilde{A} = \begin{cases} \det A & \text{if } d_i - d_j \text{ is even} \\ -\det A & \text{if } d_i - d_j \text{ is odd.} \end{cases}$$

<u>0.7. Definition.</u> Let $1 \le m \le n$ be integers, let $a_1,\dots,a_m$ be column vectors of differential forms of the length n, and let $s_1,\dots,s_m$ be integers with $s_1+\dots+s_m=n$. If all s_i are non-negative, then we set

$$\det(\overset{s_1}{\overline{a_1}},\dots,\overset{s_m}{\overline{a_m}}) = \det(\underbrace{a_1,\dots,a_1}_{s_1 \text{ times}},\dots,\underbrace{a_m,\dots,a_m}_{s_m \text{ times}}),$$

and if at least one of the integers s_i is negative, then we write

$$\det(\overset{s_1}{\overline{a_1}},\dots,\overset{s_m}{\overline{a_m}}) = 0.$$

<u>0.8. The forms $\omega(u)$ and $\omega'(v)$.</u> Let X be a real C^1 manifold, and let $u=(u_1,\dots,u_n)$, $v=(v_1,\dots,v_n)$ be two $\mathbb{C}^n$-valued C^1 maps defined on X. Then we set

$$\omega(u) = du_1\wedge\ldots\wedge du_n = \frac{1}{n!}\det(\overbrace{du}^{n})$$

and

$$\omega'(v) = \sum_{j=1}^{n}(-1)^{j+1}v_j dv_1\wedge\ldots\underset{j}{\wedge}\ldots\wedge dv_n = \frac{1}{(n-1)!}\det(v,\overbrace{dv}^{n-1})$$

on X, where $\underset{j}{\wedge}$ means that dv_j must be omitted.

If X, Y are two real C^1 manifolds and the points in X and Y are denoted by x and y, respectively, then the exterior differential operator with respect to x on $X\times Y$ will be denoted by d_x, the exterior differential operator with respect to y on $X\times Y$ will be denoted by d_y, and the "full" exterior differential operator d on $X\times Y$ then will be denoted also by $d_{x,y}$. Analogously, we use the notations $\omega_x(u)$, $\omega'_x(v)$, $\omega_{x,y}(u)$, $\omega'_{x,y}(v)$, and, if X, Y are complex manifolds, the notations ∂_x, $\bar\partial_x$, $\partial_{x,y}$, $\bar\partial_{x,y}$.

0.9. Proposition. If X is a real C^1 manifold, $v:X\longrightarrow\mathbb{C}^n$ is a C^1 map, and $F:X\longrightarrow\mathbb{C}$ is a C^1 function, then $\omega'(Fv) = F^n\omega'(v)$.

Proof. By Proposition 0.6 (i) and (ii) we have

$$\det(Fv,\overbrace{d(Fv)}^{n-1}) = \det(Fv,\overbrace{(dF)v+Fdv}^{n-1}) = \det(Fv,\overbrace{Fdv}^{n-1}) = F^n(v,\overbrace{dv}^{n-1}). \;\square$$

0.10. Proposition. If X is a real C^1 manifold and u, v are two $\mathbb{C}^n$-valued C^1 maps on X, then the form

$$\frac{\omega'(v)\wedge\omega(u)}{\langle v,u\rangle^n} = \omega'\left(\frac{v}{\langle v,u\rangle}\right)\wedge\omega(u)$$

(this equation holds true by Proposition 0.9) is closed for all $x\in X$ with $\langle v(x),u(x)\rangle \neq 0$. In particular, if moreover $\langle v,u\rangle=1$ on X, then $\omega'(v)\wedge\omega(u)$ is closed on X.

Proof. We can assume that $\langle v,u\rangle = 1$ on X (otherwise we have to replace v by $v/\langle v,u\rangle$). Then $0 = d\langle v,u\rangle = \sum v_i du_i + \sum u_i dv_i$. Hence then the forms $dv_1,\ldots,dv_n,du_1,\ldots,du_n$ are linearly dependent, and therefore $d\omega'(v)\wedge\omega(u) = \omega(v)\wedge\omega(u) = 0$. $\square$

0.11. Spaces of differential forms. Here we collect the definitions of some spaces of differential forms which will be often used in this monograph. Let X be a complex manifold, and $M\subseteq X$ a subset which is contained in the closure of the set of inner points of M.

For any differential form f on M, we set (cf. Sect. 0.4 for the definition of the Riemannian norm $\|f(z)\|$)

$$\|f\|_{0,M} = \sup_{z\in M} \|f(z)\| \tag{0.5}$$

and

$$\|f\|_{\alpha,M} = \|f\|_{0,M} + \sup_{z,x\ M} \frac{\|f(z)-f(x)\|}{|z-x|^{\alpha}} \qquad \text{if } 0<\alpha<1. \tag{0.6}$$

If $0<\alpha<1$, then a form f on M is called <u>α-Hőlder continuous</u> on M if

$$\|f\|_{\alpha,K} < \infty \qquad \text{for all compact sets } K \subseteq M. \tag{0.7}$$

The notion of a <u>C^{α} form</u> (resp. of a <u>form of class C^{α}</u>) on M will be used for any $0\leq\alpha\leq\infty$, where:

C^0 stands for "continuous";

for $k=1,2,\ldots,\infty$, we say f is a <u>C^k form on M</u> if f is a C^k form in the interior of M such that all derivatives of order $<k+1$ of f admit a continuous extension onto M;

if $0<\alpha<1$ and k is a non-negative integer, then we say f is a <u>$C^{k+\alpha}$ form</u> on M if f is a C^k form on M such that all derivatives of order $\leq k$ of f are α-Hőlder continuous on M.

We use the following notations:

$L_*^{\infty}(M)$ is the space of all <u>bounded</u> differential forms on M. Notice that if (and only if) M is relatively compact in X, then $L_*^{\infty}(M)$ does not depend on the choice of the Riemannian norm on X;

$C_*^{\alpha}(M)$ is the space of all C^{α} forms on M $(0\leq\alpha\leq\infty)$;

$Z_*^{\alpha}(M)$ is the subspace of all $f\in C_*^{\alpha}(M)$ with $\bar{\partial}f=0$ in the interior of M $(0\leq\alpha\leq\infty)$;

$E_*^{\beta\to\alpha}(M)$ is the subspace of all $f\in Z_*^{\alpha}(M)$ such that $f=\bar{\partial}u$ for some $u\in C^{\beta}(M)$ $(0\leq\alpha,\beta\leq\infty)$;

$E_*^{\alpha}(M) := E_*^{\alpha\to\alpha}(M)$ $(0\leq\alpha\leq\infty)$.

If A is one of the symbols L^{∞}, C^{α}, Z^{α}, $E^{\beta\to\alpha}$ or E^{α}, then $A_m(M)$ is the subspace of all forms of degree m in $A_*(M)$ $(1\leq m\leq 2n)$, and $A_{s,r}(M)$ is the subspace of all (s,r)-forms in $A_*(M)$ $(1\leq s,r\leq 2n)$.

By $C^{\alpha}\mathcal{O}(M)$ $(0\leq\alpha\leq\infty)$ we denote the space of all complex-valued C^{α} functions on M which are holomorphic in the interior of M. If M is open, then $\mathcal{O}(M)$ is the space of all holomorphic functions on M.

If B is an arbitrary space of differential forms on M, then by $[B]_0$