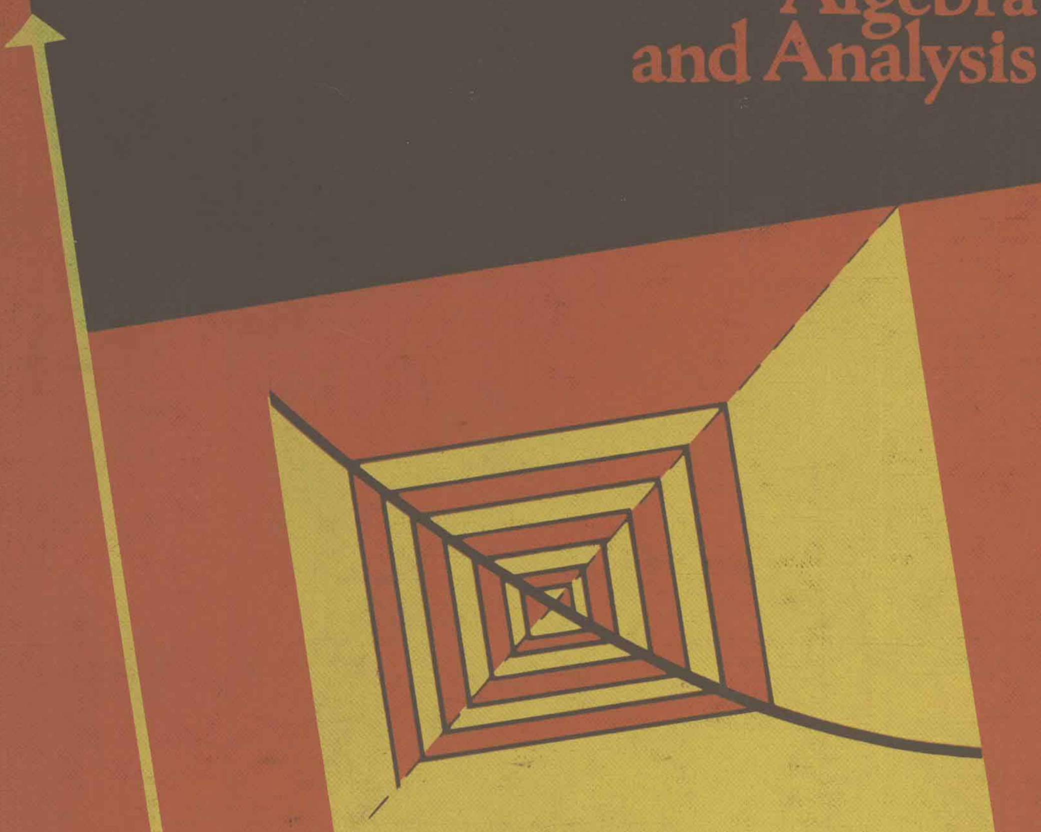
MATHEMATICAL FOUNDATIONS IN ENGINEERING AND SCIENCE

Algebra and Analysis



ANTHONY N. MICHEL / CHARLES J. HERGET

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Algebra and Analysis

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To Leone Lucille (A.N.M.)
To Marlene Jean (C.J.H.)

PREFACE

This book has evolved from a one year sequence of courses offered by the authors at Iowa State University during the past nine years. The audience for this book typically includes first or second year graduate students in various areas in engineering and science. We believe that it is also suitable for self study or as a reference book. The prerequisites include the usual background in undergraduate mathematics offered to students majoring in engineering or in the sciences. Thus, this book is suitable for advanced senior undergraduate students as well.

The objectives of this book are to (1) provide the reader with appropriate mathematical background for graduate study in engineering or science, (2) provide the reader with appropriate prerequisites for more advanced subjects in mathematics, (3) allow the student in engineering or science to become familiar with essential background in modern mathematics in an *efficient* manner, (4) give the reader a unified overview of modern mathematics, thus enabling him or her to choose additional courses in mathematics more intelligently, and, most importantly, (5) make it possible for the student to understand at an early stage of his or her graduate study the mathematics used in the current literature (journal articles and the like).

The book may be viewed as consisting essentially of three parts: set theory (Chapter 1), algebra (Chapters 2–4), and analysis (Chapters 5–7). Chapter 1 is a prerequisite for all subsequent chapters. Chapter 2 emphasizes abstract algebra (semigroups, groups, rings, etc.) and may essentially be skipped by those who are not interested in this topic. Chapter 3, which addresses linear spaces and linear transformations, is a prerequisite for Chapters 4, 6, and 7. Chapter 4, which treats finite-dimensional vector spaces and linear transformations on such spaces (matrices) is required for Chapters 6 and 7. In Chapter 5, metric spaces are treated. This chapter is a prerequisite for the subsequent chapters. Chapters 6 and 7 consider Banach and Hilbert spaces and linear operators on such spaces, respectively. Selected topics in applications, which may be omitted without loss of continuity, are presented at the ends of Chapters 2, 4, 5, 6, and 7 and include topics dealing with ordinary differential equations, integral equations, applications of the contraction mapping principle, minimization of functionals, an example from optimal control, estimation of random variables, and the like. Because of this flexibility, this book can be used as a two semester course. A comfortable pace can be established by deleting appropriate parts, taking into account the students' backgrounds and interests.

All exercises are an integral part of the text and are given when they arise rather than at the end of each chapter. Their intent is to further the reader's understanding of the subject matter.

Concerning the labeling of items, some comments are in order. Sections are assigned numerals which reflect the chapter and section numbers. For example, Section 2.3 signifies the third section in the second chapter. Extensive sections are usually divided into subsections identified by upper case common letters A, B, C, etc. Equations, definitions, theorems, corollaries, lemmas, examples, exercises, figures, and special remarks are assigned monotonically increasing numerals which identify the chapter, section, and item number. For example, Theorem 4.4.7 denotes the seventh identified item in the fourth section of Chapter 4. This theorem is followed by Eq. (4.4.8), the eighth identified item in the same section. Within a given chapter, figures are identified by upper case letters, A, B, C, etc., while outside of the chapter the same figure is identified by the above numbering scheme. Finally, the end of a proof or an example is signified by the symbol ■.

We acknowledge the contributions of the students who used the class notes which served as precursors to this book. We would like to thank Professors N. R. Amundson and R. Aris from the University of Minnesota, Professor J. A. Heinen from Marquette University, and Professors S. E. Dickson, W. R. Madych, R. K. Miller and D. L. Isaacson from Iowa State University for their valuable advice during the preparation of this manuscript. Likewise, thanks are due to Professor W. B. Boast, former Department Head of the Electrical Engineering Department at Iowa State University, and Professor J. O. Kopplin, Chairman of the Electrical Engineering Department at Iowa State University, for their support and encouragement. We are particularly appreciative of the efforts of Mrs. Betty A. Carter and Miss Shellie Siders in typing the manuscript. Above all, we would like to thank our wives, Leone and Marlene, for their patience and understanding.

A. N. Michel
C. J. Herget

CONTENTS

1

FUNDAMENTAL CONCEPTS

In this chapter we present fundamental concepts required throughout the remainder of this book. We begin by considering sets in Section 1.1. In Section 1.2 we discuss functions; in Section 1.3 we introduce relations and equivalence relations; and in Section 1.4 we concern ourselves with operations on sets. In Section 1.5 we give a brief indication of the types of mathematical systems which we will consider in this book. The chapter concludes with a brief discussion of references.

1.1. SETS

Virtually every area of modern mathematics is developed by starting from an undefined object called a **set.** There are several reasons for doing this. One of these is to develop a mathematical discipline in a completely axiomatic and totally abstract manner. Another reason is to present a unified approach to what may seem to be highly diverse topics in mathematics. Our reason is the latter, for our interest is not in abstract mathematics for its own sake. However, by using abstraction, many of the underlying principles of modern mathematics are more clearly understood.

Thus, we begin by assuming that a **set** is a well defined collection of

elements or **objects.** We denote sets by common capital letters A, B, C, etc., and elements or objects of sets by lower case letters a, b, c, etc. For example, we write

$$A = \{a, b, c\}$$

to indicate that A is the collection of elements a, b, c. If an element x belongs to a set A, we write

$$x \in A.$$

In this case we say that "x belongs to A," or "x is contained in A," or "x is a member of A," etc. If x is any element and if A is a set, then we assume that one knows whether x belongs to A or whether x does not belong to A. If x does not belong to A we write

$$x \notin A.$$

To illustrate some of the concepts, we assume that the reader is familiar with the set of real numbers. Thus, if we say

R is the set of all real numbers,

then this is a well defined collection of objects. We point out that it is possible to characterize the set of real numbers in a purely abstract manner based on an axiomatic approach. We shall not do so here.

To illustrate a non-well defined collection of objects, consider the statement "the set of all tall people in Ames, Iowa." This is clearly not precise enough to be considered here.

We will agree that any set A may not contain any given element x more than once unless we explicitly say so. Moreover, we assume that the concept of "order" will play no role when representing elements of a set, unless we say so. Thus, the sets $A = \{a, b, c\}$ and $B = \{c, b, a\}$ are to be viewed as being exactly the same set.

We usually do not describe a set by listing every element between the curly brackets $\{\ \ \}$ as we did for set A above. A convenient method of characterizing sets is as follows. Suppose that for each element x of a set A there is a statement $P(x)$ which is either true or false. We may then define a set B which consists of all elements $x \in A$ such that $P(x)$ is true, and we may write

$$B = \{x \in A : P(x) \text{ is true}\}.$$

For example, let A denote the set of all people who live in Ames, Iowa, and let B denote the set of all males who live in Ames. We can write, then,

$$B = \{x \in A : x \text{ is a male}\}.$$

When it is clear which set x belongs to, we sometimes write $\{x : P(x) \text{ is true}\}$ (instead of, say, $\{x \in A : P(x) \text{ is true}\}$).

It is also necessary to consider a set which has no members. Since a set is determined by its elements, there is only one such set which is called the

empty set, or the **vacuous set,** or the **null set,** or the **void set** and which is denoted by $\varnothing$. Any set, A, consisting of one or more elements is said to be **non-empty** or **non-void**. If A is non-void we write $A \neq \varnothing$.

If A and B are sets and if every element of B also belongs to A, then we say that B is a **subset** of A or A **includes** B, and we write $B \subset A$ or $A \supset B$. Furthermore, if $B \subset A$ and if there is an $x \in A$ such that $x \notin B$, then we say that B is a **proper subset** of A. Some texts make a distinction between proper subset and any subset by using the notation $\subset$ and $\subseteq$, respectively. We shall not use the symbol $\subseteq$ in this book. We note that if A is any set, then $\varnothing \subset A$. Also, $\varnothing \subset \varnothing$. If B is not a subset of A, we write $B \not\subset A$ or $A \not\supset B$.

1.1.1. Example. Let R denote the set of all real numbers, let Z denote the set of all integers, let J denote the set of all positive integers, and let Q denote the set of all rational numbers. We could alternately describe the set Z as

$$Z = \{x \in R: x \text{ is an integer}\}.$$

Thus, for every $x \in R$, the statement x is an integer is either true or false. We frequently also specify sets such as J in the following obvious manner,

$$J = \{x \in Z: x = 1, 2, \ldots\}.$$

We can specify the set Q as

$$Q = \left\{x \in R: x = \frac{p}{q}, p, q \in Z, q \neq 0\right\}.$$

It is clear that $\varnothing \subset J \subset Z \subset Q \subset R$, and that each of these subsets are proper subsets. We note that $0 \notin J$. ■

We now wish to state what is meant by equality of sets.

1.1.2. Definition. Two sets, A and B, are said to be **equal** if $A \subset B$ and $B \subset A$. In this case we write $A = B$. If two sets, A and B, are not equal, we write $A \neq B$. If x and y denote the same element of a set, we say that they are equal and we write $x = y$. If x and y denote distinct elements of a set, we write $x \neq y$.

We emphasize that all definitions are "if and only if" statements. Thus, in the above definition we should actually have said: A and B are equal if and only if $A \subset B$ and $B \subset A$. Since this is always understood, *hereafter all definitions will imply the "only if" portion.* Thus, we simply say: two sets A and B are said to be equal if $A \subset B$ and $B \subset A$.

In Definition 1.1.2 we introduced two concepts of equality, one of equality of sets and one of equality of elements. We shall encounter many forms of equality throughout this book.

Now let X be a set and let $A \subset X$. The **complement of subset A with respect to X** is the set of elements of X which do not belong to A. We denote the complement of A with respect to X by $C_X A$. When it is clear that the complement is with respect to X, we simply say the **complement of A** (instead of the complement of A with respect to X), and simply write $A^{\sim}$. Thus, we have

$$A^{\sim} = \{x \in X : x \notin A\}. \tag{1.1.3}$$

In every discussion involving sets, we will always have a given fixed set in mind from which we take elements and subsets. We will call this set the **universal set**, and we will usually denote this set by X.

Throughout the remainder of the present section, X denotes always an arbitrary non-void fixed set.

We now establish some properties of sets.

1.1.4. Theorem. Let A, B, and C be subsets of X. Then

(i) if $A \subset B$ and $B \subset C$, then $A \subset C$;
(ii) $X^{\sim} = \varnothing$;
(iii) $\varnothing^{\sim} = X$;
(iv) $(A^{\sim})^{\sim} = A$;
(v) $A \subset B$ if and only if $A^{\sim} \supset B^{\sim}$; and
(vi) $A = B$ if and only if $A^{\sim} = B^{\sim}$.

Proof. To prove (i), first assume that A is non-void and let $x \in A$. Since $A \subset B$, $x \in B$, and since $B \subset C$, $x \in C$. Since x is arbitrary, every element of A is also an element of C and so $A \subset C$. Finally, if $A = \varnothing$, then $A \subset C$ follows trivially.

The proofs of parts (ii) and (iii) follow immediately from (1.1.3).

To prove (iv), we must show that $A \subset (A^{\sim})^{\sim}$ and $(A^{\sim})^{\sim} \subset A$. If $A = \varnothing$, then clearly $A \subset (A^{\sim})^{\sim}$. Now suppose that A is non-void. We note from (1.1.3) that

$$(A^{\sim})^{\sim} = \{x \in X : x \notin A^{\sim}\}. \tag{1.1.5}$$

If $x \in A$, it follows from (1.1.3) that $x \notin A^{\sim}$, and hence we have from (1.1.5) that $x \in (A^{\sim})^{\sim}$. This proves that $A \subset (A^{\sim})^{\sim}$.

If $(A^{\sim})^{\sim} = \varnothing$, then $A = \varnothing$; otherwise we would have a contradiction by what we have already shown; i.e., $A \subset (A^{\sim})^{\sim}$. So let us assume that $(A^{\sim})^{\sim} \neq \varnothing$. If $x \in (A^{\sim})^{\sim}$ it follows from (1.1.5) that $x \notin A^{\sim}$, and thus we have $x \in A$ in view of (1.1.3). Hence, $(A^{\sim})^{\sim} \subset A$.

We leave the proofs of parts (v) and (vi) as an exercise. ■

1.1.6. Exercise. Prove parts (v) and (vi) of Theorem 1.1.4.

The proofs given in parts (i) and (iv) of Theorem 1.1.4 are intentionally quite detailed in order to demonstrate the exact procedure required to prove containment and equality of sets. Frequently, the manipulations required to prove some seemingly obvious statements are quite long. It is suggested that the reader carry out all the details in the manipulations of the above exercise and the exercises that follow.

Next, let A and B be subsets of X. We define the **union** of sets A and B, denoted by $A \cup B$, as the set of all elements that are in A or B; i.e.,

$$A \cup B = \{x \in X : x \in A \text{ or } x \in B\}.$$

When we say $x \in A$ or $x \in B$, we mean x is in either A or in B or in both A and B. This *inclusive* use of "or" is standard in mathematics and logic.

If A and B are subsets of X, we define their **intersection** to be the set of all elements which belong to both A and B and denote the intersection by $A \cap B$. Specifically,

$$A \cap B = \{x \in X : x \in A \text{ and } x \in B\}.$$

If the intersection of two sets A and B is empty, i.e., if $A \cap B = \varnothing$, we say that A and B are **disjoint**.

For example, let $X = \{1, 2, 3, 4, 5\}$, let $A = \{1, 2\}$, let $B = \{3, 4, 5\}$, let $C = \{2, 3\}$, and let $D = \{4, 5\}$. Then $A^{\sim} = B$, $B^{\sim} = A$, $D \subset B$, $A \cup B = X$, $A \cap B = \varnothing$, $A \cup C = \{1, 2, 3\}$, $B \cap D = D$, $A \cap C = \{2\}$, etc.

In the next result we summarize some of the important properties of union and intersection of sets.

1.1.7. Theorem. Let A, B, and C be subsets of X. Then

(i) $A \cap B = B \cap A$;
(ii) $A \cup B = B \cup A$;
(iii) $A \cap \varnothing = \varnothing$;
(iv) $A \cup \varnothing = A$;
(v) $A \cap X = A$;
(vi) $A \cup X = X$;
(vii) $A \cap A = A$;
(viii) $A \cup A = A$;
(ix) $A \cup A^{\sim} = X$;
(x) $A \cap A^{\sim} = \varnothing$;
(xi) $A \cap B \subset A$;
(xii) $A \cap B = A$ if and only if $A \subset B$;
(xiii) $A \subset A \cup B$;
(xiv) $A = A \cup B$ if and only if $B \subset A$;
(xv) $(A \cap B) \cap C = A \cap (B \cap C)$;
(xvi) $(A \cup B) \cup C = A \cup (B \cup C)$;
(xvii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$;

(xviii) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$;

(xix) $(A \cup B)^{\sim} = A^{\sim} \cap B^{\sim}$; and

(xx) $(A \cap B)^{\sim} = A^{\sim} \cup B^{\sim}$.

Proof. We only prove part (xviii) of this theorem, again as an illustration of the manipulations involved. We will first show that $(A \cap B) \cup C \subset (A \cup C) \cap (B \cup C)$, and then we show that $(A \cap B) \cup C \supset (A \cup C) \cap (B \cup C)$.

Clearly, if $(A \cap B) \cup C = \varnothing$, the assertion is true. So let us assume that $(A \cap B) \cup C \neq \varnothing$, and let x be any element of $(A \cap B) \cup C$. Then $x \in A \cap B$ or $x \in C$. Suppose $x \in A \cap B$. Then x belongs to both A and B, and hence $x \in A \cup C$ and $x \in B \cup C$. From this it follows that $x \in (A \cup C) \cap (B \cup C)$. On the other hand, let $x \in C$. Then $x \in A \cup C$ and $x \in B \cup C$, and hence $x \in (A \cup C) \cap (B \cup C)$. Thus, if $x \in (A \cap B) \cup C$, then $x \in (A \cup C) \cap (B \cup C)$, and we have

$$(A \cap B) \cup C \subset (A \cup C) \cap (B \cup C). \tag{1.1.8}$$

To show that $(A \cap B) \cup C \supset (A \cup C) \cap (B \cup C)$ we need to prove the assertion only when $(A \cup C) \cap (B \cup C) \neq \varnothing$. So let x be any element of $(A \cup C) \cap (B \cup C)$. Then $x \in A \cup C$ and $x \in B \cup C$. Since $x \in A \cup C$, then $x \in A$ or $x \in C$. Furthermore, $x \in B \cup C$ implies that $x \in B$ or $x \in C$. We know that either $x \in C$ or $x \notin C$. If $x \in C$, then $x \in (A \cap B) \cup C$. If $x \notin C$, then it follows from the above comments that $x \in A$ and also $x \in B$. Then $x \in A \cap B$, and hence $x \in (A \cap B) \cup C$. Thus, if $x \notin C$, then $x \in (A \cap B) \cup C$. Since this exhausts all the possibilities, we conclude that

$$(A \cup C) \cap (B \cup C) \subset (A \cap B) \cup C. \tag{1.1.9}$$

From (1.1.8) and (1.1.9) it follows that $(A \cup C) \cap (B \cup C) = (A \cap B) \cup C$. ■

1.1.10. Exercise. Prove parts (i) through (xvii) and parts (xix) and (xx) of Theorem 1.1.7.

In view of part (xvi) of Theorem 1.1.7, there is no ambiguity in writing $A \cup B \cup C$. Extending this concept, let n be any positive integer and let $A_1, A_2, \ldots, A_n$ denote subsets of X. The set $A_1 \cup A_2 \cup \ldots \cup A_n$ is defined to be the set of all $x \in X$ which belong to at least one of the subsets A_i, and we write

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n = \{x \in X : x \in A_i \text{ for some } i = 1, \ldots, n\}.$$

Similarly, by part (xv) of Theorem 1.1.7, there is no ambiguity in writing $A \cap B \cap C$. We define

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n = \{x \in X : x \in A_i \text{ for all } i = 1, \ldots, n\}.$$