ANALYSIS ON REAL AND COMPLEX MANIFOLDS

Raghavan NARASIMHAN

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EDITORIAL NOTE

Advanced Studies in Pure Mathematics consists of monographs and expository texts. It will cover important recent developments in contemporary mathematics, as well as topics for which a satisfactory systematic modern treatment is lacking. The editors hope that the existence of this series will tempt potential authors to fill some of the many gaps in the literature which are being felt by an increasing number of workers in various mathematical fields. Advanced Studies in Pure Mathematics will include books in English and French, and, possibly, translations.

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Preface

This book has its origin in lectures given at the Tata Institute of Fundamental Research, Bombay in the winter of 1964/65. The aim of the lectures was to present various topics in analysis, both on real and on complex manifolds. It is unnecessary to add that the topics actually chosen were determined entirely by personal taste. The contents were issued as lecture notes by the Tata Institute, and the present book is based on these notes.

The book is meant for people interested in analysis, who have little analytical background. The elements of the theory of functions of real variables (differential and integral calculus and measure theory) and some complex variable theory are assumed. Elementary properties of functions of several complex variables which are used are, in general, stated explicitly with references. It is however supposed that the reader is well acquainted with linear and multilinear algebra (properties of duals, tensor products, exterior products and so on of vector spaces) as well as set topology (properties of connected and locally compact spaces). (The material required is contained in Bourbaki: *Algèbre Lineaire*, *Algèbre Multilinéaire*, and *Topologie Générale*, *Chap*. I & II).

There are three chapters. The first deals with properties of differentiable functions in R^n. The aim is to present, with complete proofs, some theorems on differentiable functions which are often used in differential topology (such as the implicit function theorem, Sard's theorem and Whitneys' approximation theorem).

The second chapter is meant as an introduction to the study of real and complex manifolds. Apart from the usual definitions (differential forms and vector fields) this chapter contains an exposition of the theorem of Frobenius, the lemmata of Poincaré and Grothendieck with applications of Grothendieck's lemma to complex analysis, the imbedding theorem of Whitney and Thom's transversality theorem.

The last chapter deals with properties of linear elliptic differential operators. Characterizations of linear differential operators, due to Peetre and to Hörmander are given. The inequalities of Gårding and of Friedrichs on elliptic operators are proved and are used to prove the regularity of weak solutions of elliptic equations. The chapter ends with the approximation theorem of Malgrange–Lax and its application to the proof of the Runge theorem on open Riemann surfaces due to Behnke and Stein.

We have not dealt with Riemannian metrics and elementary differential geometry. Nor have we dealt with elliptic complexes in spite of their importance and interest. It is actually not very difficult to extend the theorems, such as the finiteness theorem of Chap. 3, to such complexes.

It remains for me to acknowledge the help I have received in preparing this book. My thanks are due to Mrs. M. Narlikar who wrote the notes issued by the Tata Institute; I am specially indebted to H. G. Diamond who read, very carefully, a large part of these notes, pointed out mistakes, and suggested improvements and different proofs. Finally, I am grateful to N. H. Kuiper for his invitation to rewrite the Tata Institute notes as a book, for his helpful remarks on Chapters 1 and 2 and for his assistance in preparing the manuscript for the printer.

Genève, July 1968.

Raghavan Narasimhan

Contents

CHAPTER 1

Differentiable functions in R^n

Notation. We shall use the following notation. We use R, C, Q, Z to denote, respectively, the field of real numbers, the field of complex numbers, the field of rational numbers and the ring of integers. We shall look upon the first two as being provided with their usual topology. R^n, C^n, . . . will denote the Cartesian product of R, C, . . ., respectively, so that, e.g.,

$$R^n = \{(x_1, \ldots, x_n) |\, x_j \in R, j = 1, \ldots, n\}.$$

The notations R^+, Q^+, Z^+ stand for the sets of non-negative elements of R, Q, Z respectively.

For the most part, α, β stand for n-tuples of non-negative integers, $\alpha=(\alpha_1, \ldots, \alpha_n)$, $\beta=(\beta_1, \ldots, \beta_n)$, α_j, $\beta_j \in Z^+$. We then set

$$|\alpha| = \alpha_1 + \ldots + \alpha_n, \qquad \alpha! = \alpha_1! \cdots \alpha_n!,$$

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!} \quad \text{if} \quad \beta_j \leqq \alpha_j.$$

We write $\beta \leqq \alpha$ if $\beta_j \leqq \alpha_j$ and $\beta < \alpha$ if $\beta \leqq \alpha$ and $\beta \neq \alpha$.

We denote a point of R^n [C^n] by $x = (x_1, \ldots, x_n)$ [$z = (z_1, \ldots, z_n)$]. Then

$$|x| = \max_j |x_j|, \qquad |z| = \max_j |z_j|,$$

$$\|x\| = (|x_1|^2 + \ldots + |x_n|^2)^{\frac{1}{2}}, \quad \|z\| = (|z_1|^2 + \ldots + |z_n|^2)^{\frac{1}{2}}$$

and

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \qquad z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}.$$

If X is a (Hausdorff) topological space and S a subset of X, we denote by $\mathring{S}$ the interior of S, i.e., the largest open set contained in S. If S_1, S_2 are two subsets of X, we write $S_1 \Subset S_2$ if S_1 is relatively compact in S_2; i.e., if the closure of S_1 in S_2 is compact.

If f is a map of an open set Ω in R^n into R^q and λ a function $\geqq 0$ in Ω, we write

$$f(x) = O(\lambda(x)), \qquad [\text{or } f = O(\lambda)]$$

if there is a constant $C > 0$ such that $|f(x)| \leqq C\lambda(x)$ for all $x \in \Omega$. In addition, if $a \in \Omega$, we write

$$f(x) = o(\lambda(x)),$$

as $x \to a$ (or $|x - a| \to 0$), if there is a map $\varepsilon\colon \Omega \to R^+$ such that $\varepsilon(x) \to 0$ as $x \to a$ and $|f(x)| \leqq \varepsilon(x)\lambda(x)$.

Similar notation is used when a is replaced by a "point at infinity".

§ 1.1 Taylor's formula

Let Ω be an open set in R^n and k an integer $\geqq 0$. We denote by $C^k(\Omega)$ the set of real-valued functions f on Ω which possess continuous partial derivatives of order $\leqq k$, i.e., for which the derivatives

$$\frac{\partial^{\alpha_1 + \ldots + \alpha_n} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

exist and are continuous on Ω for $|\alpha| = \alpha_1 + \ldots + \alpha_n \leqq k$. We denote by $C^\infty(\Omega)$ the set of functions belonging to $C^k(\Omega)$ for all $k \geqq 0$. Functions in $C^k(\Omega)$ are called C^k functions on Ω. For $f \in C^k(\Omega)$ and $|\alpha| \leqq k$, we denote the partial derivative

$$\frac{\partial^{\alpha_1 + \ldots + \alpha_n} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

by

$$D^\alpha f = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} f.$$

The order in which the differentiations are performed is irrelevant.

For any function f (not necessarily continuous) defined on Ω, we denote by supp(f) the closure in Ω of the set

$$\{x \in \Omega \mid f(x) \neq 0\}.$$

Supp(f) is called the *support* of f. $C_0^k(\Omega)$ stands for the set of $f \in C^k(\Omega)$ such that supp(f) is compact.

If E is a finite-dimensional R vector space, we denote by $C^k(\Omega, E)$, $C_0^k(\Omega, E), \ldots$ the set of all mappings $f: \Omega \to E$ such that, for any (continuous) linear functional l on E, the function $l \circ f \in C^k(\Omega)$, $C_0^k(\Omega), \ldots$.

If $e_1, \ldots, e_q$ is an R basis of E, and $f: \Omega \to E$ is a map, for each $x \in \Omega$, there are real numbers $f_1(x), \ldots, f_q(x)$ such that

$$f(x) = \sum_{j=1}^{q} f_j(x) e_j.$$

It is easily checked that

$$f \in C^k(\Omega, E), C_0^k(\Omega, E), \ldots$$

if and only if

$$f_j \in C^k(\Omega), C_0^k(\Omega), \ldots \qquad \text{for } j = 1, \ldots, q.$$

Elements of $C^k(\Omega, E)$ are called C^k mappings of Ω into E. If $E = R^q$ we write $C^k(\Omega, q)$, $C_0^k(\Omega, q), \ldots$ for $C^k(\Omega, E)$, $C_0^k(\Omega, E), \ldots$. For $f \in C^k(\Omega, E)$, we can define the derivatives $D^\alpha f$ for $|\alpha| \leq k$. Then

$$D^\alpha f \in C^{k-|\alpha|}(\Omega, E).$$

We shall identify $f \in C_0^k(\Omega)$ with the element $g \in C_0^k(R^n)$ which $= f$ on Ω and $= 0$ on $R^n - \Omega$.

We shall often deal with complex-valued functions on Ω (or mappings of Ω into C^q). We shall then use the notation $C^k(\Omega)$, $C^k(\Omega, q), \ldots$ for $C^k(\Omega, C)$, $C^k(\Omega, C^q), \ldots$ if no confusion is likely.

A real-valued function f defined on Ω is called (*real*) *analytic* if, for any $a = (a_1, \ldots, a_n) \in \Omega$, there exists a power series

$$P_a(x) \equiv \sum_\alpha c_\alpha (x-a)^\alpha = \sum_{\alpha_j \geq 0} c_{\alpha_1 \ldots \alpha_n} (x_1 - a_1)^{\alpha_1} \ldots (x_n - a_n)^{\alpha_n},$$

which converges to $f(x)$ for x in a neighbourhood U of a. The series then converges uniformly to f on compact subsets of U (so that f

is continuous) and so does the differentiated series. Hence $f \in C^\infty(\Omega)$ and for any $\beta = (\beta_1, \ldots, \beta_n)$,

$$D^\beta f(x) = D^\beta P_a(x) = \sum_\alpha c_\alpha D^\beta (x-a)^\alpha.$$

Moreover, the series is uniquely determined by f; in fact

$$c_\alpha = \frac{1}{\alpha!} D^\alpha f(a).$$

Analytic maps of Ω into a finite dimensional vector space are defined in the same way as above.

If U, V are open sets in R^n and $f: U \to V$ is a homeomorphism such that both f and f^{-1} are C^k mappings, we say that f is a C^k *diffeomorphism* (or just diffeomorphism) of U onto V. If $U = V$, we call f a C^k *automorphism*.

If f and f^{-1} are real analytic, we speak of an analytic isomorphism (or automorphism if $U = V$).

If U is an open set in C^n and f a complex-valued function on U, f is called *holomorphic* if for any $a \in U$, there is a power series $\sum c_\alpha (z-a)^\alpha$, which converges to $f(z)$ for all z in a neighbourhood of a.

If E is a finite dimensional C vector space, a map $f: U \to E$ is called *holomorphic* if for any C linear function l on E, $l \circ f$ is holomorphic. A map $f: \Omega \to C^q$ is holomorphic if and only if, when we write $f = (f_1, \ldots, f_q)$, each f_j is a holomorphic function.

A map $f: U \to V$ (open sets in C^n) is called a *C analytic isomorphism* (or, by abuse of language, an analytic isomorphism if no confusion is likely) if f and f^{-1} are holomorphic. A theorem of Osgood (*see* e.g. HERVÉ [1963]), which we shall not prove in this book, asserts that a one-one holomorphic map of U onto V is a C analytic isomorphism. There is no analogue for C^k or real analytic maps.

We shall assume some elementary properties of holomorphic functions. These are proved in most books on several complex variables, *see* e.g. HERVÉ [1963], and HÖRMANDER [1966].

1.1.1 CAUCHY-RIEMANN EQUATIONS. A function defined on an open set $U \subset C^n$ is holomorphic if and only if it is continuous and, for any j, $1 \leq j \leq n$, the partial derivatives

$$\frac{\partial f}{\partial \bar{z}_j} \equiv \frac{1}{2}\left(\frac{\partial f}{\partial x_j} + \mathrm{i}\,\frac{\partial f}{\partial y_j}\right)$$

exist and are 0. Here $z_j = x_j + \mathrm{i}y_j$, x_j, y_j are real and $\mathrm{i} = \sqrt{-1}$.

We also set

$$\frac{\partial f}{\partial z_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} - \mathrm{i}\,\frac{\partial f}{\partial y_j}\right).$$

For a *holomorphic* function f on U, we write

$$D^\alpha f = \left(\frac{\partial}{\partial z_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial z_n}\right)^{\alpha_n} f.$$

In view of the equations 1.1.1, we have

$$D^\alpha f = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} f.$$

A basic theorem of Hartogs (*see* HÖRMANDER [1966]) asserts that the condition of continuity is superfluous in the Cauchy-Riemann-equations 1.1.1; we shall not prove this here.

1.1.2 PRINCIPLE OF ANALYTIC CONTINUATION. If f is holomorphic (real analytic) in a connected open set $U(\Omega)$ in C^n (R^n) and $D^\alpha f(a) = 0$ for all $\alpha = (\alpha_1, \ldots, \alpha_n)$ and some $a \in U(\Omega)$, then $f \equiv 0$. In particular, if f vanishes on a non-empty open subset of $U(\Omega)$, then $f \equiv 0$.

1.1.3 WEIERSTRASS' THEOREM. If $\{f_\nu\}$ is a sequence of holomorphic functions, converging uniformly on compact subsets of U to a function f, then f is holomorphic in U. Moreover, for any α, $\{D^\alpha f_\nu\}$ converges to $D^\alpha f$, uniformly on compact sets.

1.1.3′ MONTEL'S THEOREM. If $\mathfrak{S} = \{f\}$ is a family of holomorphic functions in U which is uniformly bounded on compact subsets K of U:

$$|f(x)| \leq M \qquad \text{for all } x \in K,\ f \in \mathfrak{S},$$

then any sequence of elements of $\mathfrak{S}$ contains a subsequence which converges uniformly on compact subsets of U.

1.1.3″ THE MAXIMUM PRINCIPLE. Let f be holomorphic in a connected open set U in C^n. Then, the map $f: U \to C$ is either constant or open. In particular, if U is bounded and we set

$$M = \sup_{\zeta \in \partial U} \overline{\lim_{z \to \zeta,\, z \in U}} |f(z)|,$$

we have $|f(z)| < M$ for all $z \in U$ unless f is constant.

1.1.4 CAUCHY'S INEQUALITIES. If f is holomorphic in U and $|f(z)| \leqq M$ for $z \in U$, then for any compact set $K \subset U$ and any α, we have

$$|D^\alpha f(z)| \leqq M\, \alpha! \delta^{-|\alpha|} \qquad \text{for } z \in K,$$

where δ is the distance of K from the boundary of U.

1.1.5 LEMMA. Let f be real analytic in $\Omega \subset R^n$. We look upon R^n as a closed subset of C^n. Then there exists an open set $U \subset C^n$, $U \cap R^n = \Omega$ and a holomorphic function F in U with $F|\Omega = f$.

PROOF. Let $a \in \Omega$ and let $P_a(x) = \sum c_\alpha (x-a)^\alpha$ be a power series converging to $f(x)$ for $|x-a| < r_a$, $r_a > 0$. Define

$$U_a = \{z \in C^n |\ |z-a| < r_a\}.$$

Then, for $z \in U_a$, $P_a(z) = \sum c_\alpha (z-a)^\alpha$ converges and is a holomorphic function on U_a.

Let $U = \bigcup_{a \in \Omega} U_a$. We assert that if $U_a \cap U_b = U_{a,b} \neq \emptyset$, then $P_a = P_b$ in $U_{a,b}$. In fact, $U_{a,b}$ is convex, hence connected. Further, if $U_{a,b} \neq \emptyset$, then $U_{a,b} \cap R^n \neq \emptyset$ and, for any $c \in U_{a,b} \cap R^n$, we have

$$D^\alpha P_a(c) = D^\alpha f(c) = D^\alpha P_b(c),$$

and we may apply principle 1.1.2. Hence we may define a holomorphic function F on U by setting $F|U_a = P_a$. Clearly $F|\Omega = f$.

We return now to real valued functions. Let N be a neighbourhood of the closed unit interval $0 \leqq t \leqq 1$ in R, and let $f \in C^k(N)$, $k \geqq 1$. Then we have:

1.1.6 LEMMA. There is a ξ with $0 \leqq \xi \leqq 1$ such that

$$f(1) = \sum_{\nu=0}^{k-1} \frac{f^{(\nu)}(0)}{\nu!} + \frac{f^{(k)}(\xi)}{k!},$$

where

$$f^{(\nu)}(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\nu} f(t).$$

PROOF. For a continuous function g, on N, set

$$I_0(g, t) = g(t), \qquad I_r(g, t) = \int_0^t I_{r-1}(g, s)\,\mathrm{d}s, \qquad r \geqq 1.$$

Clearly, if $g \in C^k(N)$ and $g^{(\nu)}(0) = 0$ for $0 \leqq \nu \leqq k-1$, we have

$$g(t) = I_k(g^{(k)}, t).$$

If we apply this to

$$g(t) = f(t) - \sum_{\nu=0}^{k-1} \frac{f^{(\nu)}(0)}{\nu!} t^{\nu},$$

we obtain

1.1.7 $$f(1) - \sum_{\nu=0}^{k-1} \frac{f^{(\nu)}(0)}{\nu!} = I_k(g^{(k)}, 1) = I_k(f^{(k)}, 1).$$

If m and M denote respectively the lower and upper bounds of $f^{(k)}$ in [0, 1], we have, clearly,

$$\frac{m}{k!} \leqq I_k(f^{(k)}, 1) \leqq \frac{M}{k!}.$$

Since $f^{(k)}$ is continuous, and so takes all values between m and M, there is a ξ, $0 \leqq \xi \leqq 1$ for which

$$I_k(f^{(k)}, 1) = \frac{1}{k!} f^{(k)}(\xi).$$

This proves the lemma.

It is easy to prove by induction that

$$I_k(g,t) = \frac{1}{(k-1)!}\int_0^t g(s)(t-s)^{k-1}\,\mathrm{d}s.$$

Hence, (1.1.7) can be written in the form: