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Groups and Analysis

Edited by
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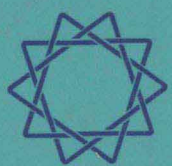

Groups and Analysis

The legacy of Hermann Weyl

Edited by

KATRIN TENT

Universität Bielefeld

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Preface

This volume grew out of the conference in honour of Hermann Weyl that took place in Bielefeld in September 2006.

Weyl was born in 1885 in Elmshorn, a small town near Hamburg. He studied mathematics in Göttingen and Munich, and obtained his doctorate in Göttingen under the supervision of Hilbert. After taking a teaching post for a few years, he left Göttingen for Zürich to accept a Chair of Mathematics at the ETH Zürich, where he was a colleague of Einstein just at the time when Einstein was working out the details of the theory of general relativity. Weyl left Zürich in 1930 to become Hilbert's successor at Göttingen, moving to the new Institute for Advanced Study in Princeton, New Jersey after the Nazis took power in 1933. He remained there until his retirement in 1951. Together with his wife, he spent the rest of his life in Princeton and Zürich, where he died in 1955.

The Collaborative Resarch Centre (SFB 701) *Spectral Structures and Topological Methods in Mathematics* has manifold connections with the areas of mathematics that were founded or influenced by Weyl's work. These areas include geometric foundations of manifolds and physics, topological groups, Lie groups and representation theory, harmonic analysis and analytic number theory as well as foundations of mathematics.

In 1913, Weyl published *Die Idee der Riemannschen Fläche* ('The Concept of a Riemann Surface'), giving a unified treatment of Riemann surfaces.

He described the development of relativity theory in his *Raum, Zeit, Materie* ('Space, Time, Matter') from 1918, which reached a fourth edition in 1922. In 1918, he introduced the concept of gauge and gave the first example of what is now known as a gauge theory.

From 1923 to 1938, Weyl developed the theory of compact groups in terms of matrix representations and proved a fundamental character formula for compact Lie groups. His book *Classical Groups* opened new directions in invariant theory. It covered symmetric groups, general

linear groups, orthogonal groups, and symplectic groups, and results on their invariants and representations.

In *The Continuum*, Weyl developed the logic of classical analysis along the lines of Brouwer's intuitionism. However, he later decided that this radical constructivism puts too much of a restriction on his mathematics and reconciled himself with the more formalistic ideas of Hilbert.

Weyl also showed how to use exponential sums in diophantine approximation, with his criterion for uniform distribution modulo one, which was a fundamental contribution to analytic number theory.

During the conference, his lasting influence on current mathematics became evident through a series of impressive talks often connecting theorems of Weyl with the most current results in dynamical systems, invariant theory, or partial differential equations. We are happy that so many speakers agreed to contribute to this volume.

The conference was funded by the Collaborative Research Center (SFB 701) 'Spectral structures and topological methods in mathematics'. We gratefully acknowledge support by the German Research Foundation (DFG). Thanks are also due to Philip Herrmann for editing this volume, and to Markus Rost and Ulf Rehmann.

Bielefeld, December 2007

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1

Harmonic Analysis on Compact Symmetric Spaces: the Legacy of Elie Cartan and Hermann Weyl

Roe Goodman

Department of Mathematics
Rutgers, The State University of New Jersey

1 Introduction

In his lecture *Relativity theory as a stimulus in mathematical research* [Wey4], Hermann Weyl says that "Frobenius and Issai Schur's spadework on finite and compact groups and Cartan's early work on semi-simple Lie groups and their representations had nothing to do with it [relativity theory]. But for myself I can say that the wish to understand what really is the mathematical substance behind the formal apparatus of relativity theory led me to the study of representations and invariants of groups, and my experience in this regard is probably not unique."

Weyl's first encounter with Lie groups and representation theory as a tool to understand relativity theory occurred in connection with the Helmholtz-Lie space problem and the problem of decomposing the tensor product $\otimes^k \mathbb{C}^n$ under the mutually commuting actions of the general linear group $\mathrm{GL}(n,\mathbb{C})$ (on each copy of $\mathbb{C}^n$) and the symmetric group $\mathfrak{S}_k$ (in permuting the k copies of $\mathbb{C}^n$).[1] He later described the tensor decomposition problem in general terms [Wey3] as "an epistemological principle basic for all theoretical science, that of projecting the actual upon the background of the possible." Mathematically, the issue was to find subspaces of tensor space that are invariant and irreducible under all transformations that commute with $\mathfrak{S}_k$. This had already been done by Frobenius and Schur around 1900, but apparently Weyl first became aware of these results in the early 1920's. The subspaces in question, which are the ranges of minimal projections in the group algebra of $\mathfrak{S}_k$, are exactly the irreducible (polynomial) representations of $\mathrm{GL}(n,\mathbb{C})$, and all irreducible representations arise this way for varying k by including multiplication by integral powers of $\det(g)$ in the action. It seems clear

1 see [Haw, §11.2-3]

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from his correspondence with Schur at this time that these results were Weyl's starting point for his later work in representation theory and invariant theory.

Near the end of his monumental paper on representations of semisimple Lie groups [Wey1, Kap. IV, §4], Weyl considers the problem of constructing all the irreducible representations of a simply-connected simple Lie group G such as $\mathrm{SL}(n,\mathbb{C})$. This had been done on a case-by-case basis by Cartan [Car1], starting with the defining representations for the classical groups (or the adjoint representation for the exceptional groups) and building up a general irreducible representation by forming tensor products. By contrast, Weyl, following the example of Frobenius for finite groups, says that "the correct starting point for building representations does not lie in the adjoint group, but rather in the *regular representation*, which through its reduction yields *in one blow* all irreducible representations." He introduces the infinite-dimensional space $C(U)$ of all continuous functions on the compact real form U of G ($U = \mathrm{SU}(n)$ when $G = \mathrm{SL}(n,\mathbb{C})$) and the right translation representation of U on $C(U)$. He then obtains the irreducible representations of U and their characters by using the eigenspaces of compact integral operators given by left convolution with positive-definite functions in $C(U)$, in analogy with the decomposition of tensor spaces for $\mathrm{GL}(n,\mathbb{C})$ using elements of the group algebra of $\mathfrak{S}_k$. The details are spelled out in the famous Peter–Weyl paper [Pe-We], which proves that the normalized matrix entries of the irreducible unitary representations of U furnish an orthonormal basis for $L^2(U)$, and that every continuous function on U is a uniform limit of linear combinations of these matrix entries.

In the introduction to [Car2], É. Cartan says that his paper was inspired by the paper of Peter and Weyl, but he points out that for a compact Lie group their use of integral equations "gives a transcendental solution to a problem of an algebraic nature" (namely, the completeness of the set of finite-dimensional irreducible representations of the group). Cartan's goal is "to give an algebraic solution to a problem of a transcendental nature, more general than that treated by Weyl." Namely, to find an explicit decomposition of the space of all L^2 functions on a homogeneous space into an orthogonal direct sum of group-invariant irreducible subspaces.

Cartan's paper [Car2] then stimulated Weyl [Wey2] to treat the same problem again and write "the systematic exposition by which I should like to replace the two papers Peter–Weyl [Pe-We] and Cartan [Car2]." In his characteristic style of finding the core of a problem through gen-

eralization, Weyl takes the finite-dimensional irreducible subspaces of functions (which he calls the *harmonic sets* by analogy with the case of spherical harmonics) on the compact homogeneous space X as his starting point.[2] Using the invariant measure on the homogenous space, he constructs integral operators that intertwine the representation of the compact group U on $C(X)$ with the left regular representation on $C(U)$.

In this paper we approach the Weyl–Cartan results by way of algebraic groups. The *finite* functions on a homogeneous space for a compact connected Lie group (that is, the functions whose translates span a finite-dimensional subspace) can be viewed as *regular* functions on the complexified group (a complex reductive algebraic group). Irreducible subspaces of functions under the action of the compact group correspond to irreducible subspaces of regular functions on the complex reductive group—this is Weyl's *unitarian trick*. We describe the algebraic group version of the Peter–Weyl decomposition and geometric criterion for simple spectrum of a homogeneous space (due to E. Vinberg and B. Kimelfeld). We present R. Richardson's algebraic group version of the Cartan embedding of a symmetric space, and the celebrated results of Cartan and S. Helgason concerning finite-dimensional spherical representations.

We then turn to more recent results of J.-L. Clerc [Cle] concerning the complexified Iwasawa decomposition and zonal spherical functions on a compact symmetric space, and S. Gindikin's construction ([Gin1], [Gin2], [Gin3]) of the *horospherical Cauchy–Radon transform*, which shows that compact symmetric spaces have canonical dual objects that are complex manifolds.

We make frequent citations to the extraordinary books of A. Borel [Bor] and T. Hawkins [Haw], which contain penetrating historical accounts of the contributions of Weyl and Cartan. Borel's book also describes the development of algebraic groups by C. Chevalley that is basic to our approach. For a survey of other developments in harmonic analysis on symmetric spaces from Cartan's paper to the mid 1980's see Helgason [Hel3]. Thanks go to the referee for pointing out some notational inconsistencies and making suggestions for improving the organization of this paper.

2 Weyl's emphasis on function spaces, rather than the underlying homogeneous space, is in the spirit of the recent development of *quantum groups*; his immediate purpose was to make his theory sufficiently general to include also J. von Neumann's theory of almost-periodic functions on groups, in which the functions determine a compactification of the underlying group.

2 Algebraic Group Version of Peter–Weyl Theorem

2.1 Isotypic Decomposition of $\mathcal{O}[X]$

The paper [Pe-We] of Peter and Weyl considers compact Lie groups U; because the group is compact left convolution with a continuous function is a compact operator. Hence such an operator, if self-adjoint, has finite-dimensional eigenspaces that are invariant under right translation by elements of U. The finiteness of the invariant measure on U also guarantees that every finite-dimensional representation of U carries a U-invariant positive-definite inner product, and hence is *completely reducible* (decomposes as the direct sum of irreducible representations).[3]

Turning from Weyl's transcendental methods to the more algebraic and geometric viewpoint preferred by Cartan, we recall that a subgroup $G \subset \mathrm{GL}(n, \mathbb{C})$ is an *algebraic group* if it is the zero set of a collection of polynomials in the matrix entries. The *regular functions* $\mathcal{O}[G]$ are the restrictions to G of polynomials in matrix entries and $\det^{-1}$. In particular, G is a complex Lie group and the regular functions on G are holomorphic. A finite-dimensional complex representation (π, V) of G is *rational* if the matrix entries of the representation are regular functions on G. The group G is *reductive* if every rational representation is completely reducible.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. From the work of Cartan, Weyl, and Chevalley, one knows the following:

(1) There is a simply-connected complex linear algebraic group G with Lie algebra $\mathfrak{g}$.

(2) The finite-dimensional representations of $\mathfrak{g}$ correspond to rational representations of G.

(3) There is a real form $\mathfrak{u}$ of $\mathfrak{g}$ and a simply-connected compact Lie group $U \subset G$ with Lie algebra $\mathfrak{u}$.

(4) The finite-dimensional unitary representations of U extend uniquely to rational representations of G, and U-invariant subspaces correspond to G-invariant subspaces.[4]

(5) The irreducible rational representations of G are parameterized by the positive cone in a lattice of rank l (Cartan's theorem of the *highest weight*).[5]

3 This is the Hurwitz "trick" (*Kunstgriff*) that Weyl learned from I. Schur; see Hawkins [Haw, §12.2].

4 This is Weyl's *unitary trick*.

5 The first algebraic proofs of this that did not use case-by-case considerations were found by Chevalley and Harish-Chandra in 1948; see [Bor, Ch. VII, §3.6-7].

The highest weight construction is carried out as follows: Fix a Borel subgroup $B = HN^+$ of G (a maximal connected solvable subgroup). Here $H \cong (\mathbb{C}^\times)^l$, with $l = \text{rank}(G)$, is a maximal algebraic torus in G, and N^+ is the unipotent radical of B associated with a set of positive roots of H on $\mathfrak{g}$. Let $\bar{B} = HN^-$ be the opposite Borel subgroup. We can always arrange the embedding $G \subset \mathrm{GL}(n, \mathbb{C})$ so that H consists of the diagonal matrices in G, N^+ consists of the upper-triangular unipotent matrices in G, and N^- consists of the lower-triangular unipotent matrices in G. Let $\mathfrak{h}$ be the Lie algebra of H and $\Phi \subset \mathfrak{h}^*$ the roots of $\mathfrak{h}$ on $\mathfrak{g}$. Write $P(\Phi) \subset \mathfrak{h}^*$ for the *weight lattice* of H and $P_{++} \subset P(\Phi)$ for the *dominant weights*, relative to the system of positive roots determined by N^+. For $\lambda \in P(\Phi)$ we denote by $h \mapsto h^\lambda$ the corresponding character of H. It extends to a character of B by $(hn)^\lambda = h^\lambda$ for $h \in H$ and $n \in N^+$.

An irreducible rational representation (π, E) of G is then determined (up to equivalence) by its *highest weight*. The subspace E^{N^+} of N^+-fixed vectors in E is one-dimensional, and H acts on it by a character $h \mapsto h^\lambda$ where $\lambda \in P_{++}$. The subspace E^{N^-} of N^--fixed vectors in E is also one-dimensional, and H acts on it by the character $h \mapsto h^{-\lambda^*}$ where $\lambda^* = -w_0 \cdot \lambda$. Here w_0 is the element of the Weyl group of $(\mathfrak{g}, \mathfrak{h})$ that interchanges positive and negative roots.

For each $\lambda \in P_{++}$ we fix a model (π_λ, E_λ) for the irreducible rational representation with highest weight λ. Then $(\pi_{\lambda^*}, E_\lambda^*)$ is the contragredient representation. Fix a highest weight vector $e_\lambda \in E_\lambda$ and a lowest weight vector $f_{\lambda^*} \in E_\lambda^*$, normalized so that

$$\langle e_\lambda, f_{\lambda^*} \rangle = 1.$$

Here we are using $\langle v, v^* \rangle$ to denote the tautological duality pairing between a vector space and its dual (in particular, this pairing is complex linear in both arguments). For dealing with matrix entries as regular functions on the complex algebraic group G this is more convenient than using a U-invariant inner product on E_λ and identifying E_λ^* with E_λ via a conjugate-linear map.

Let X be an irreducible affine algebraic G space. Denote the regular functions on X by $\mathcal{O}[X]$. There is a representation ρ of G on $\mathcal{O}[X]$:

$$\rho(g)f(x) = f(g^{-1}x) \quad \text{for } f \in \mathcal{O}[X] \text{ and } g \in G.$$

Because the G-action is algebraic, $\mathrm{Span}\{\rho(G)f\}$ is a finite-dimensional rational G-module for $f \in \mathcal{O}[X]$. There is a tautological G-intertwining map

$$E_\lambda \otimes \mathrm{Hom}_G(E_\lambda, \mathcal{O}[X]) \to \mathcal{O}[X],$$

given by $v \otimes T \mapsto Tv$. For $\lambda \in P_{++}$ let

$$\mathcal{O}[X]^{N^+}(\lambda) = \{f \in \mathcal{O}[X] : \rho(hn)f = h^\lambda f \quad \text{for } h \in H \text{ and } n \in N^+\}. \tag{2.1}$$

The key point is that the choice of a highest weight vector e_λ gives an isomorphism

$$\mathrm{Hom}_G(E_\lambda, \mathcal{O}[X]) \cong \mathcal{O}[X]^{N^+}(\lambda). \tag{2.2}$$

Here a G-intertwining map T applied to the highest weight vector gives the function $\varphi = Te_\lambda \in \mathcal{O}[X]^{N^+}(\lambda)$, and conversely every such function φ defines a unique intertwining map T by this formula.[6] From (2.2) we see that the highest weights of the G-irreducible subspaces of $\mathcal{O}[X]$ comprise the set

$$\mathrm{Spec}(X) = \{\lambda \in P_{++} : \mathcal{O}[X]^{N^+}(\lambda) \neq 0\} \quad (\text{the } G \textit{ spectrum} \text{ of } X)$$

Using the isomorphism (2.2) and the reductivity of G, we obtain the decomposition of $\mathcal{O}[X]$ under the action of G, as follows:

Theorem 2.1 *The isotypic subspace of type (π_λ, E_λ) in $\mathcal{O}[X]$ is the linear span of the G-translates of $\mathcal{O}[X]^{N^+}(\lambda)$. Furthermore,*

$$\mathcal{O}[X] \cong \bigoplus_{\lambda \in \mathrm{Spec}(X)} E_\lambda \otimes \mathcal{O}[X]^{N^+}(\lambda) \quad \textit{(algebraic direct sum)} \tag{2.3}$$

as a G-module, with action $\pi_\lambda(g) \otimes 1$ on the λ summand.

The action of G on $\mathcal{O}[X]$ is not only linear; it also preserves the algebra structure. Since $\mathcal{O}[X]^{N^+}(\lambda) \cdot \mathcal{O}[X]^{N^+}(\mu) \subset \mathcal{O}[X]^{N^+}(\lambda+\mu)$ under pointwise multiplication and $\mathcal{O}[X]$ has no zero divisors (X is irreducible), it follows from (2.3) that

$$\mathrm{Spec}(X) \text{ is an additive subsemigroup of } P_{++}.$$

The multiplicity of π_λ in $\mathcal{O}[X]$ is $\dim \mathcal{O}[X]^{N^+}(\lambda)$ (which may be infinite). All of this was certainly known (perhaps in less precise form) by Cartan and Weyl at the time [Pe-We] appeared. We now consider Cartan's goal in [Car2] to determine the decomposition (2.3) when G acts transitively on X; especially, when X is a symmetric space. This requires determining the *spectrum* and the *multiplicities* in this decomposition.

6 Weyl uses a similar construction in [Wey2], defining intertwining maps by integration over a compact homogeneous space.

2.2 Multiplicity Free Spaces

We say that an irreducible affine G-space X is *multiplicity free* if all the irreducible representations of G that occur in $\mathcal{O}[X]$ have multiplicity one. Thanks to the theorem of the highest weight, this property can be translated into a geometric statement (see [Vi-Ki]). For a subgroup $K \subset G$ and $x \in X$ write $K_x = \{k \in L : k \cdot x = x\}$ for the isotropy group at x.

Theorem 2.2 (Vinberg–Kimelfeld) *Suppose there is a point $x_0 \in X$ such that $B \cdot x_0$ is open in X. Then X is multiplicity free. In this case, if $\lambda \in \mathrm{Spec}(X)$ then $h^\lambda = 1$ for all $h \in H_{x_0}$.*

Proof If $B \cdot x_0$ is open in X, then it is Zariski dense in X (since X is irreducible). Hence $f \in \mathcal{O}[X]^{N^+}(\lambda)$ is determined by $f(x_0)$, since on the dense set $B \cdot x_0$ it satisfies $f(b \cdot x_0) = b^{-\lambda} f(x_0)$. In particular, if $f \neq 0$ then $f(x_0) \neq 0$, and hence $h^\lambda = 1$ for all $h \in H_{x_0}$. Thus

$$\dim \mathcal{O}[X]^{N^+}(\lambda) \leq 1 \quad \text{for all } \lambda \in P_{++}.$$

Now apply Theorem 2.1. □

Remark. The converse to Theorem 2.2 is true; this depends on some results of Rosenlicht [Ros] and is the starting point for the classification of multiplicity free spaces (see [Be-Ra]).

Example: Algebraic Peter–Weyl Decomposition

Theorem 2.2 implies the algebraic version of the Peter-Weyl decomposition of the regular representation of G. Consider the reductive group $G \times G$ acting on $X = G$ by left and right translations. Denote this representation by ρ:

$$\rho(y, z) f(x) = f(y^{-1} x z), \quad \text{for} \quad f \in \mathcal{O}[G] \quad \text{and} \quad x, y, z \in G.$$

Take $H \times H$ as the Cartan subgroup and $\bar{B} \times B$ as the Borel subgroup of $G \times G$. Let $x_0 = I$ (the identity in G). The orbit of x_0 under the Borel subgroup is

$$(\bar{B} \times B) \cdot x_0 = N^- H N^+ \quad \text{(Gauss decomposition)} \tag{2.4}$$

This orbit is open in G since $\mathfrak{g} = \mathfrak{n}^- + \mathfrak{h} + \mathfrak{n}^+$. Hence G is multiplicity free as a $G \times G$ space. The $G \times G$ highest weights (relative to this choice of Borel subgroup) are pairs $(w_0 \mu, \lambda)$, with $\lambda, \mu \in P_{++}$. The diagonal

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