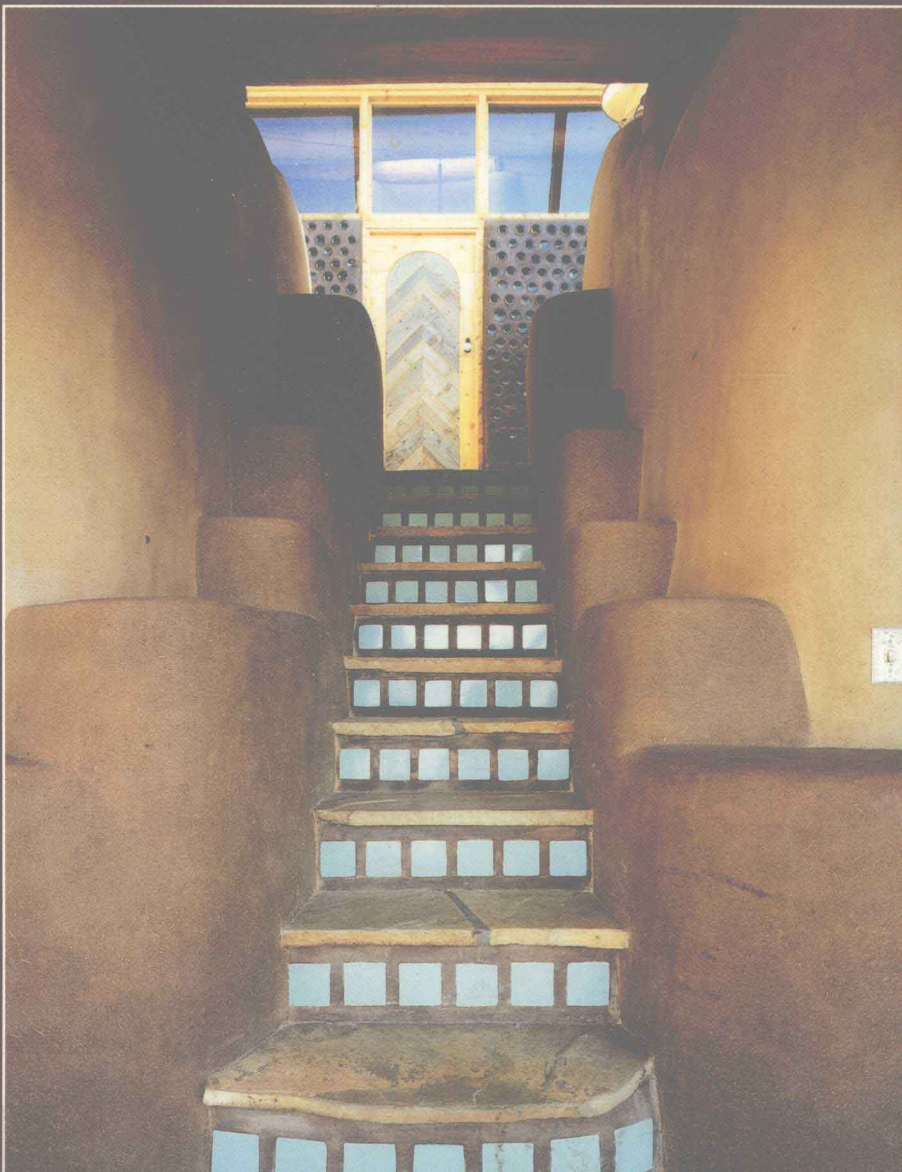
MATHEMATICAL THINKING

PROBLEM-SOLVING AND PROOFS



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Mathematical Thinking

Problem-Solving and Proofs

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To all who enjoy mathematical puzzles,
and to our loved ones,
who tolerate our enjoyment of them.

Preface for the Student

This book aims to be both enjoyable and demanding. We present interesting problems and develop the basic undergraduate mathematics needed to solve them. Below we list 36 such problems. We solve most of these in this book, while at the same time developing enough theory to prepare you for upper-division math courses.

This course will differ from other math courses you have taken, because it emphasizes writing and language skills. We do not ask that you memorize formulas, but rather that you learn to express yourself clearly and accurately. You will learn to solve mathematical puzzles as well as to write proofs of theorems from elementary algebra, discrete mathematics, and calculus. This will broaden your knowledge and improve the clarity of your thinking.

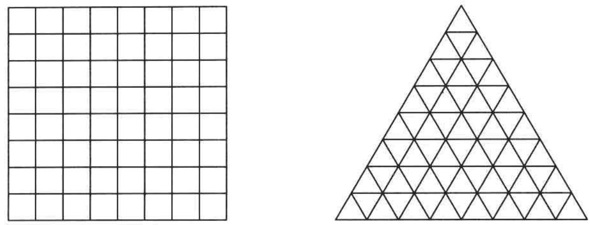
How can you improve your writing? Good writing requires practice. Rereading and revising solutions can improve your presentation. You must say what you mean and mean what you say. Mathematics offers a tremendous opportunity for this, because clear decisions can be made about whether sentences contain faulty reasoning. Mathematics uses formulas to express complicated thoughts, and you will learn how to combine well chosen notation with clear explanation in sentences. This will enable you to communicate ideas concisely and accurately.

We invite you to consider some intriguing problems. Solutions to most appear in the text, and we include the others as exercises.

1. Given several piles of pennies, we create a new collection by removing one coin from each old pile to make one new pile. Each original pile shrinks by one; 1,1,2,5 becomes 1,4,4, for example. Which lists of sizes (order is unimportant) are unchanged under this operation?

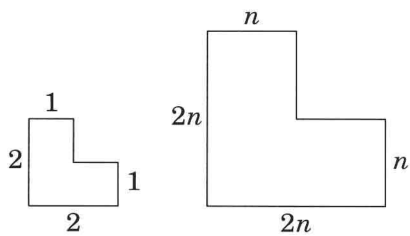
2. Which natural numbers are sums of consecutive smaller natural numbers? For example, $30 = 9 + 10 + 11$ and $31 = 15 + 16$, but 32 has no such representation.

3. Including squares of all sizes (one-by-one through eight-by-eight), an ordinary eight-by-eight checkerboard has 204 squares. How many squares of all sizes arise using an n-by-n checkerboard? How many triangles of all sizes arise using a triangular grid with sides of length n?



4. At a party with five married couples, no person shakes hands with his or her spouse. Of the nine people other than the host, no two shake hands with the same number of people. With how many people does the hostess shake hands?

5. Is it possible to fill the large region below with non-overlapping copies of the small L-shape? Rotations and translations are allowed.



6. We can tell whether two collections of weights have the same total weight by placing them on a balance scale. How many known weights are needed to balance each integer weight from 1 to 121? How should these weights be chosen? (Known weights can be placed on either side or omitted.)

7. If each resident of New York City has 100 coins in a jar, is it possible that no two residents have the same number of coins of each type (pennies, nickels, dimes, quarters, half-dollars)?

8. How can we find the greatest common divisor of two large numbers without factoring them?

9. Why are there infinitely many prime numbers? Why are there arbitrarily long stretches of consecutive non-prime positive integers?

10. Consider a dart board having two regions, one worth a points and the other worth b points, where a and b are positive integers having no common factors. What is the largest point total that cannot be obtained by throwing darts at the board?

11. A math professor cashes a check for x dollars and y cents, but the teller inadvertently pays y dollars and x cents. After the professor buys a newspaper for k cents, the remaining money is twice as much as the original value of the check. If $k = 50$, what was the value of the check? If $k = 75$, why is this situation impossible?

12. Must there be a Friday the 13th in every year?

13. When two digits in the base 10 representation of an integer are interchanged, the difference between the old number and the new number is divisible by nine. Why?

14. A positive integer is *palindromic* if reversing the digits of its base 10 representation doesn't change the number. Why is every palindromic integer with an even number of digits divisible by 11? What happens in other bases?

15. How can one describe all the integer solutions to $42x + 63y = z$, or to $x^2 + y^2 = z^2$?

16. Suppose L is a prime number. For which positive integers K can we express the rational number K/L as the sum of the reciprocals of two positive integers?

17. Are there more rational numbers than integers? Are there more real numbers than rational numbers? What does "more" mean for infinite sets?

18. Can player A have a higher batting average than player B in day games and in night games but a lower batting average than player B over all games?

Player	Day	Night	Overall
A	.333	.250	.286
B	.300	.200	.290

19. Suppose A and B gamble as follows: On each play, each player shows 1 or 2 fingers, and one pays the other x dollars, where x is the total number of fingers showing. If x is odd, then A pays B; if x is even, then B pays A. Who has the advantage, and how can that player exploit it?

20. Given a positive integer k, how can we obtain a formula for the sum $1^k + 2^k + \cdots + n^k$?

21. Suppose candidates A and B in an election receive a and b votes, respectively. If the votes are counted in a random order, what is the probability that candidate A never trails?

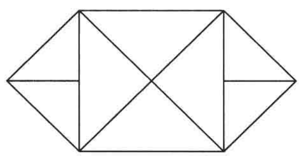
22. Can the numbers $0, \ldots, 100$ be written in some order so that no 11 positions contain numbers that successively increase or successively decrease? (An increasing or decreasing set need not occupy consecutive positions or use consecutive numbers.)

23. Suppose each dot in an n by n grid of dots is colored black or white. How large must n be to guarantee the existence of a rectangle whose corners have the same color?

24. How many positive integers less than 1,000,000 have no common factors with 1,000,000?

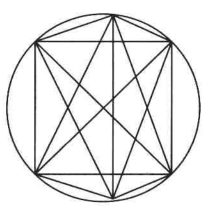
25. Suppose n students take an exam, and the exam papers are handed back at random for peer grading. What is the probability that no student gets his or her own paper back? What happens to this probability as n goes to infinity?

26. A computer plotter is to draw a figure on a page. How can one determine the minimum number of times the pen must be lifted while drawing the figure?



27. Suppose there are n girls and n boys at a party, and each girl likes some of the boys. Under what conditions is it possible to pair the girls with boys so that each girl is paired with a boy that she likes?

28. Suppose n points lie on a circle. How many regions are created by drawing all chords joining these points, assuming that no three chords have a common intersection?



29. A Platonic solid has congruent regular polygons as faces and has the same number of faces meeting at each vertex. Why are the tetrahedron, cube, octahedron, dodecahedron, and icosahedron the only ones?

30. Suppose n spaces are available for parking along the side of a street. We can fill the spaces using Rabbits, which take one space, and/or Cadillacs, which take two spaces. In how many ways can we fill the spaces? In other words, how many lists of 1's and 2's sum to n?

31. Repeatedly pushing the "x^2" button on a calculator generates a sequence tending to 0 if the initial positive value is less than 1 and tending to ∞ if it is greater than 1. What happens with other quadratic functions?

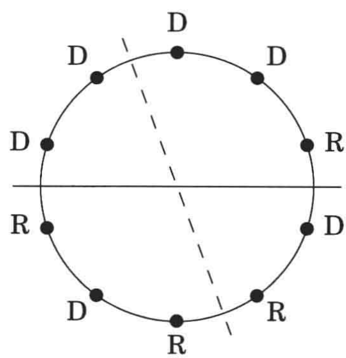
32. What numbers have more than one decimal representation?

33. Suppose that the points in a tennis game are independent and that the server wins each point with probability p. What is the probability that the server wins the game?

34. How is $\lim_{n\to\infty}(1 + x/n)^n$ relevant to compound interest?

35. One type of baseball player hits singles with probability p and otherwise strikes out. Another type hits home runs with probability $p/4$ and otherwise strikes out. Assume that a single advances each runner by two bases. Compare a team composed of the home-run hitters with a team composed of the singles hitters. Which team generates more runs per inning?

36. Suppose two jewel thieves steal a circular necklace with $2m$ gold beads and $2n$ silver beads arranged in some unknown order. Why is it that, for any arrangement of the beads, there is a way to cut the necklace along some diameter so that each thief gets half the beads of each color? Why is it that a heated circular wire always contains two diametrically opposite points where the temperature is the same? How are these questions related?



Preface for the Instructor

This book arose from discussions about the undergraduate mathematics curriculum. We asked several questions. Why do students find it difficult to write proofs? What is the role of discrete mathematics? How can the curriculum better integrate diverse topics? Perhaps most important, why don't students enjoy and appreciate mathematics as much as we might hope?

Upper-division courses in mathematics expose serious gaps in the preparation of students; the difficulties are particularly evident in elementary real analysis courses. Such courses present two obstacles to students. First, the concepts of elementary analysis are subtle; it took mathematicians centuries to understand limits. Second, proofs require both attention to exposition and a different intellectual attitude from computation. The combination of the two difficulties has defeated many students. Basic courses in linear or abstract algebra evince similar difficulties and can be overly formal. Due to their specialized focus, upperclass courses often fail to address adequately the need for careful exposition. If students first learn techniques of proof and habits of careful exposition, then they will better appreciate more advanced mathematics when they encounter it.

The excitement of mathematics springs from engaging problems. Students have natural mathematical curiosity about problems such as those listed in the Preface for the Student. They then care about the techniques used to solve them; hence we use these problems as a focus of development. We hope that students and instructors will enjoy this approach as much as we have.

A course introducing techniques of proof should not specialize in one area of mathematics; later courses offer ample opportunities for specialization. This book considers diverse problems and demonstrates relationships among several areas of mathematics. One of the authors studies complex analysis in several variables, the other studies discrete mathematics. We explored the interactions between discrete and continuous mathematics to create a course on problem-solving and proofs.

Content

We present elementary aspects of algebra and number theory, combinatorics, and analysis. We develop such diverse topics as prime factorization, modular arithmetic, Pythagorean triples, techniques of counting, basic graph theory, recurrence relations, sequences and series, the basic theorems of calculus, continuous nowhere differentiable functions, and the fundamental theorem of algebra. We integrate these topics into a coherent whole, choosing material that illustrates techniques of proof and interactions among the topics.

Part I (Elementary Concepts) begins by deriving the quadratic formula and using it to motivate the axioms for the real numbers, which we agree to assume. We discuss sets, logical statements, and functions, paying careful attention to the use of language. The highlight of Part I is the application of induction to several engaging problems.

Part II (Properties of Numbers) studies the number systems $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$. We explore q-ary expansions, cardinality, binomial coefficients, the Euclidean algorithm, and prime factorization. Equivalence relations provide the foundation for our development of modular arithmetic and the rational numbers. Features include the Schroeder-Bernstein Theorem, Fermat's Little Theorem, the Chinese Remainder Theorem, criteria for irrationality, Pythagorean triples, Simpson's Paradox, and a bit of probability.

Part III (Discrete Mathematics) explores combinatorial arguments. We consider elementary enumeration, the pigeonhole principle, the inclusion-exclusion principle, graphs, and recurrence relations. Highlights include Bertrand's Ballot Problem (Catalan numbers), more on probability, the Euler totient function, Hall's Theorem on systems of distinct representatives, Platonic solids, and the Fibonacci numbers. Combinatorial problems lead us to recurrence relations and sequences. We develop various techniques to solve recurrences. Familiarity with sequences facilitates the transition to continuous mathematics.

Part IV (Continuous Mathematics) begins with the Least Upper Bound Property for the real numbers. We prove the Bolzano-Weierstrass Theorem and use it to prove that Cauchy sequences converge. We then develop the theory of calculus: sequences, series, continuity, differentiation, uniform convergence, and the Riemann integral. We define the natural logarithm via integration and the exponential function via infinite series, and we prove their inverse relationship. We define trigonometric functions via infinite series, using results on interchange of limiting operations to verify their properties. We include Cantor's proof that $\mathbb{R}$ is uncountable, convexity, and van der Waarden's example of a continuous and nowhere differentiable function. We omit many applications covered adequately in calculus courses, such as Taylor polynomials, analytic geometry, Kepler's laws, polar coordinates, and many of the physical interpretations of derivatives and integrals. We close by developing the properties of complex numbers and proving the Fundamental Theorem of Algebra.

In Appendix A we develop the properties of arithmetic and construct the real number system using Cauchy sequences. There we begin with $\mathbb{N}$ and subsequently construct $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$. We include a portion of the construction of $\mathbb{Q}$ in Chapter 8 in order to illustrate the fundamental role of equivalence relations.

Pedagogy

Certain pedagogical issues require careful attention. In order to benefit from this course, students must have a sense of intellectual progress. An axiomatic treatment of the real numbers seems painfully slow and frustrates students. They have learned algebraic computational techniques throughout their schooling, and we want to build on this foundation. This dictates our starting point. In Chapter 1 we list the axioms for the real numbers and their elementary algebraic consequences, and we accept them for computation and reasoning. We defer the construction of the real numbers and verification of the field axioms to Appendix A, for later appreciation. We generally do not assign the exercises in Chapter 1 that request verification of algebraic properties from the axioms.

Chapters 2 and 3 discuss elementary aspects of quantifiers, mathematical language, and functions. This material provides the language for all subsequent discussion. Formal discussion of mathematical language is problematic; students master techniques of proofs through examples of usage, not via memorization of terminology from formal logic. Thus we do not stress the formal manipulation of logical symbols. After the discussion in Chapter 2 that emphasizes the *use* of logic, familiarity with logical concepts is conveyed by repeated usage throughout the book. Chapter 2 can be treated lightly in class; students can refer to it when they need help with manipulating logical statements.

The collection of exercises is a strong feature of this book. Many are fun, some are routine, and some are difficult. Exercises designated by "(−)" or "(+)" are routine or difficult, respectively; those designated by "(!)" are especially interesting or instructive. Having used these designations, we order the exercises for each chapter roughly in parallel to the presentation of material in the text, rather than in order of difficulty. Most exercises emphasize thinking and writing rather than computation. The understanding and communication of mathematics through problem-solving should be the driving force of the course.

The Preface to the Student lists many engaging problems. Some of these begin chapters as motivating "Problems"; others are left to the exercises. Solutions of such problems in the text are designated as "Solutions". Items designated as "Examples" are generally easier than those designated as "Solutions" or "Applications". "Examples" serve primarily to illustrate concepts, whereas "Solutions" or "Applications" employ the concepts being developed and involve additional reasoning. Items designated as "Remarks" contain important commentary for which the designations of "Definition" and "Lemma" are not appropriate; students should not ignore the "Remarks". Many of the exercises in the text carry hints; these represent what we feel will be helpful to most students. We also provide an appendix with more elementary hints for many problems; these are meant to help students get started.

This book does not assume calculus and hence in principle can be used in a course taught to freshmen or to high school students. It does require motivation and commitment from the students, since problems can no longer be solved by plugging numbers into a template. The book should be appropriate for students who have studied calculus computationally and wonder why the computations work. It is ideal for beginning majors in mathematics and computer science. Readers outside mathematics who enjoy careful thinking and are curious about mathematics will also profit by it. High school teachers of mathematics may appreciate the interaction between problem-solving and theory.

Design of Courses

We developed this book through numerous courses, beginning with a version we team-taught in 1991 at the University of Illinois. Various one-semester courses can be constructed from this material. A one-semester course on discrete mathematics that emphasizes proofs will cover Parts I-III, omitting most of Chapter 8 (rational numbers) and the more algebraic material from Chapters 6 and 7. A one-semester course in elementary analysis covers Chapters 4 and 5, part of Chapter 8, and Part IV. The full text is suitable for a one-year course culminating in the Fundamental Theorem of Algebra. Using Part I and a selection of material from Parts II-IV, we have taught one-semester courses introducing students to proofs and to a balanced overview of mathematics.

The book offers considerable flexibility in the design of a balanced course. We cover almost all of Chapters 1-6, including the Schroeder-Bernstein Theorem. In Chapter 7, we skip the section on groups. From Chapter 8, the construction of rational numbers and the existence of irrational numbers are indispensable, but the proof of the former can be treated lightly. The material on Pythagorean triples and on probability in Chapters 8 and 9 enriches the course if time permits. In Chapter 9, the material up to the Summation Identity (and the ballot problem) should be covered to illuminate combinatorial reasoning. Although optional, the treatment of multinomial coefficients helps unify the course. In Chapter 10, the pigeonhole principle illustrates the elegance of concise arguments; inclusion-exclusion via the totient function further enhances cohesiveness. Chapter 11 contains enough material for the instructor to sample according to taste and time. In Chapter 12, most instructors will want to discuss first-order and second-order recurrences and perhaps the Catalan recurrence. Chapters 13 through 15 should be covered thoroughly. Coverage in Chapter 16 depends on the instructor's interests and the students' abilities; reaching the Mean Value Theorem probably requires skipping Chapter 11. The rest can be left as supplemental reading for the interested student. The one-semester balanced course will not reach Chapters 17 and 18.

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