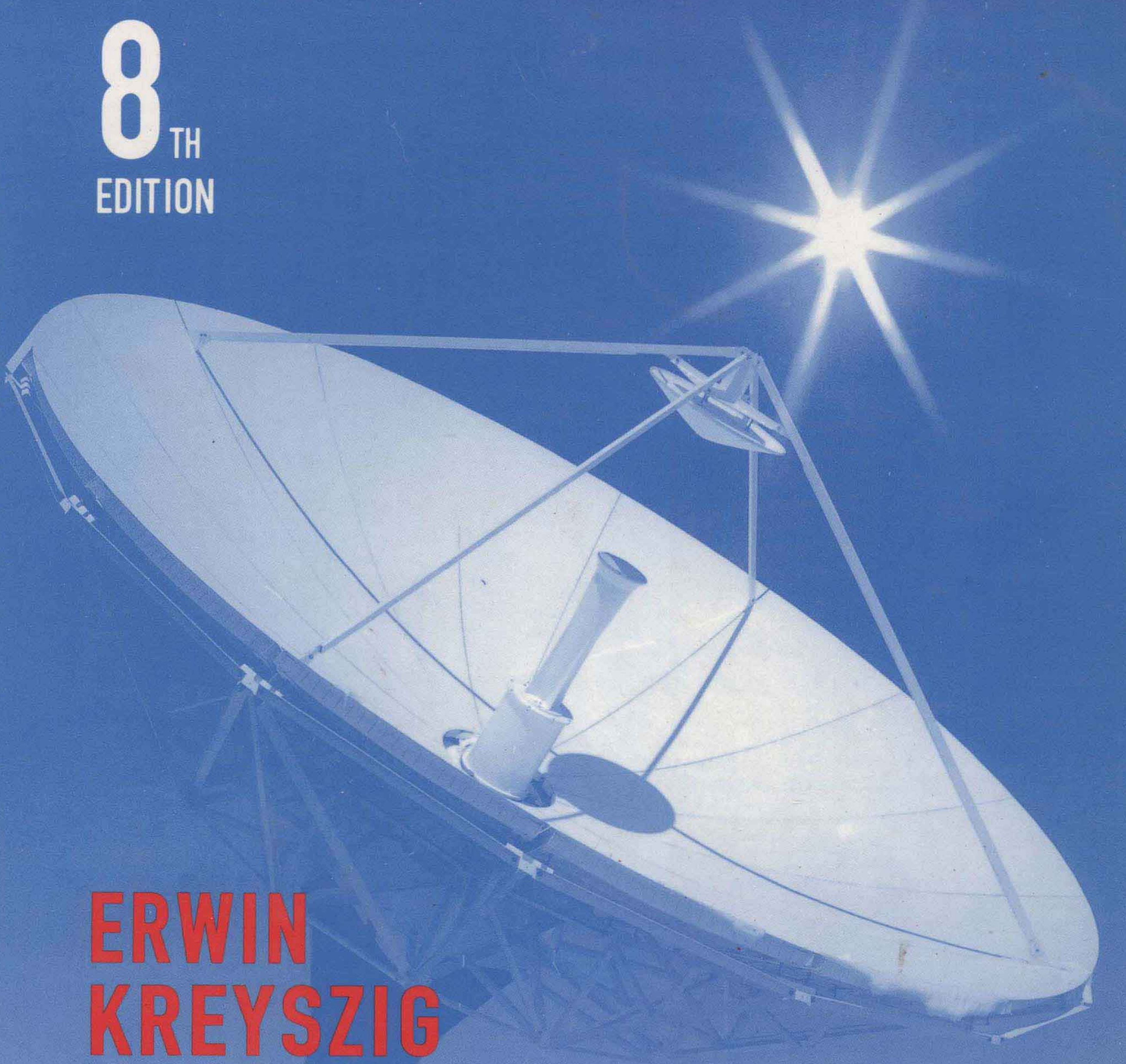
8TH EDITION



ERWIN KREYSZIG

ADVANCED ENGINEERING MATHEMATICS

STUDENT SOLUTIONS MANUAL

HERBERT KREYSZIG
ERWIN KREYSZIG

Student Solutions Manual
to accompany

Advanced Engineering Mathematics

Eighth Edition

Herbert Kreyszig

Erwin Kreyszig
Professor of Mathematics
Ohio State University
Columbus, Ohio

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PREFACE

ADVANCED ENGINEERING MATHEMATICS (AEM), 8th Edition (New York: J. Wiley and Sons, Inc., 1999) introduces students of engineering, physics, mathematics, and computer science to those areas of mathematics which from a modern viewpoint are most important in connection with practical problems. The book consists of the following independent parts;

A Ordinary Differential Equations (Chaps. 1-5)
B Linear Algebra, Vector Calculus (Chaps. 6-9)
C Fourier Analysis, Partial Differential Equations (Chaps. 10, 11)
D Complex Analysis (Chaps. 12-16)
E Numerical Methods (Chaps. 17-19)
F Optimization, Graphs (Chaps. 20, 21)
G Probability, Statistics (Chaps. 22, 23)

AEM includes a problem set after each section of every chapter. These problem sets consist of about 4500 problems and team projects.

This **Student Solutions Manual** will enhance the effectiveness of AEM. The manual contains worked-out solutions and helpful suggestions to carefully selected odd-numbered problems (to which Appendix 2 of AEM gives only the final answers). The solutions discussed in this Manual illustrate the basic ideas, techniques, and applications of the material in the text. This will provide a representative cross-section of the main topics in the book as a learning environment for the student.

The Manual may give help in working assignments, in clarifying conceptual and technical difficulties, and in developing skills. It can also be used in reviewing the course material, in studying for exams, and for self-study.

The following two facts are important to fully understanding the role and character of this Manual;

1. Whereas the book AEM itself gives only answers to (odd-numbered) problems (in Appendix 2), ***this Manual provides complete solutions*** with all the conceptual and technical details, considered from a practical point of view.

2. In addition to those problem sets, AEM contains numerous examples in the text. It is true that these examples include worked-out solutions, but we wish to emphasize very distinctly that ***the presentation in this Manual is much more detailed and leisurely and of a substantially lower level than the level of those examples in AEM***. Thus, worked-out examples in the book and worked-out problems in this Manual belong to two different categories of objects designed to enhance understanding, intuition, and skill in two different ways and on two different levels.

For best results in using this Manual follow these rules.

RULE 1. First try to solve your problem without help. A solution obtained completely by yourself will be much more valuable to you than one obtained with outside help. The more you have to struggle, the greater will be your gain in knowledge, skill, and self-confidence. It is of minor importance whether your way of solution is shortest or most elegant; essential is that you obtain a solution at all.

RULE 2. Look at worked-out examples in the text. Find out whether one of them is similar to your problem or, at least, one or another idea in an example might be useful in solving your problem.

RULE 3. Use this Manual stepwise. That is, if you reach an impasse, find out from the Manual the next step of the solution, without looking at the further steps. This act of self-discipline is part of a process of maturing, involving great rewards for you.

RULE 4. Analyze your work critically. Find reasons for the difficulties in solving the problem, which you wish to overcome, such as lack of understanding of word problems in general, difficulties in grasping the meaning of a concept or a theorem needed, insufficient insight into the general idea on which a solution method is based, deficiencies in differentiation or integration, lack of technical skill in algebraic manipulations, and so on.

RULE 5. Find a better method. Try to design a solution method that is shorter than that given in this Manual, more elegant, logically more appealing to you, or more adequate from the viewpoint of applications.

Organize your worksheets as clearly and legibly as you can. Use letter size paper, don't use little scraps of paper. You will be surprised to see how improvement in form will entail improvement in content.

Best wishes for success and fun in using this Manual.

Acknowledgement: The authors wish to thank Professor E. J. Norminton for various valuable suggestions as well as for his great help in the design of this book.

Herbert Kreyszig
and
Erwin Kreyszig

CONTENTS

PART A. ORDINARY DIFFERENTIAL EQUATIONS

CHAPTER 1. First-Order Differential Equations

Sec. 1.1 Basic Concepts and Ideas

Comment on (5). $y = cx - c^2$, hence $y' = c$, and $y'^2 - xy' + y = c^2 - xc + (cx - c^2) = 0$.

Problem Set 1.1. Page 8

1. **Calculus.** This is a problem of calculus, namely, to integrate x^2, giving $\frac{1}{3}x^3 + c$, where the constant of integration c is arbitrary. This is essential. It means that the differential equation $y' = x^2$ has infinitely many solutions, each of these cubical parabolas corresponding to a certain value of c. Sketch some of them.

13. **Initial value problem.** $y' = -2ce^{-2x}$ by differentiation. Hence the left side becomes

$$y' + 2y = -2ce^{-2x} + 2(ce^{-2x} + 1.4) = 2.8.$$

This verifies the given solution $y = ce^{-2x} + 1.4$. For $x = 0$ you have $e^0 = 1$ and thus $y(0) = c + 1.4$, which is required to be equal to 1.0. Hence $1.0 = c + 1.4, c = -0.4$, and the answer is $y = -0.4e^{-2x} + 1.4$.

23. **Falling body.** $s = gt^2/2 = 100$ [m]. Here $g = 9.80$ m/sec^2 since s is measured in meters. Using $s = 100$ and solving for t gives

$$t = \sqrt{\frac{100}{g/2}} = 10\sqrt{\frac{1}{4.9}} = 4.52 \text{ [sec]}.$$

The second result, 6.389 sec, is less than twice the first because the motion is accelerated, the velocity increases.

Sec. 1.2 Geometrical Meaning of $y' = f(x,y)$. Direction Fields

Problem Set 1.2. Page 12

1. **Calculus.** Note that the solution curves are *not* congruent because c is a factor, not an additive constant (as, for instance, in Prob. 5).

5. **Verification of solution.** Geometrically, the solution curves are obtained from each other by translations in the y-direction; they are congruent because c is an additive constant.

7. **Verification of solution.** At each point (x, y) the tangent direction of the solution is $-x/y$, hence perpendicular to the slope y/x of the ray from (0, 0) to (x, y), suggesting that the solutions are concentric circles about the origin. You can prove this by calculus, as follows. Multiply the equation by y, obtaining $yy' = -x$. Then integrate on both sides with respect to x. This gives

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + c \quad \text{or} \quad y^2 + x^2 = 2c.$$

15. **Initial value problem.** The idea from calculus just applied in Prob. 7 here gives $(9/2)y^2 + 2x^2 = c$ or $4x^2 + 9y^2 = 2c$; these are the ellipses $x^2/9 + y^2/4 = c/18$.

17. Initial value problem. In this section the usual notation is (2), that is, $y' = f(x,y)$, and the direction field lies in the xy-plane. In Prob. 17 the equation is $v' = f(t,v) = g - bv^2/m$. Hence the direction field lies in the tv-plane. With $m = 1$ and $b = 1$ the equation becomes $v' = g - v^2$. Then $v = 3.13$ gives $g - v^2 = 9.80 - 3.13^2 = 0$, approximately. The differential equation now shows that v' must be identically zero. Conclude that $v = 3.13$ must be a solution. For $v < 3.13$ you have $v' > 0$ (increasing curves) and for $v > 3.13$ you have $v' < 0$ (decreasing curves). Note that the isoclines are the horizontal parallel straight lines $g - v^2 = const$, thus $v = const$.

Sec. 1.3 Separable Differential Equations

Problem Set 1.3. Page 18

3. General solution by separation. Dividing by the right side gives

$$\frac{y'}{1+0.01y^2} = 1 \quad \text{or} \quad \frac{dy}{1+0.01y^2} = dx. \tag{A}$$

Now integrate. This is one of the more important integrals; set $v = 0.1y$ to get $y = 10v$, $dy = 10\,dv$, and from (A),

$$10\,dv/(1+v^2) = dx, \qquad \text{integrated} \qquad 10 \arctan v = x + C.$$

Recalling that $v = 0.1y$ gives $10 \arctan 0.1y = x + C$. This implies

$$y = 10 \tan(0.1(x+C)) = 10 \tan(0.1x + c), \qquad c = 0.1C.$$

15. Initial value problem. Separate variables and integrate on both sides (by parts on the right) to get

$$dy/y^2 = 2(x+1)e^{-x}\,dx, \qquad -1/y = (-2x-4)e^{-x} + c.$$

Multiply by -1 and take the reciprocal,

$$y = 1/[(2x+4)e^{-x} - c].$$

From the initial condition $y(0) = 1/6$ obtain by setting $x = 0$

$$1/6 = y(0) = 1/(4-c), \qquad \text{hence} \qquad 6 = 4 - c, \quad c = -2.$$

Inserting this into y gives the answer.

23. Initial value problem. Dividing the given equation by x^2 and setting $y/x = u$, hence $y = xu$ and $y' = u + xu'$, gives

$$(y/x)y' = u(u + xu') = 2u^2 + 4.$$

Subtracting u^2 on both sides gives $xuu' = u^2 + 4$. Separate variables, then multiply both sides by 2, and integrate with respect to x on both sides,

$$2u\,du/(u^2+4) = 2\,dx/x, \qquad \ln(u^2+4) = \ln(x^2) + C, \qquad u^2 + 4 = cx^2.$$

Solving for u^2 and taking roots gives $y/x = u = \sqrt{cx^2 - 4}$, so that

$$y = ux = \sqrt{cx^4 - 4x^2}.$$

From this and the initial condition,

$$y(2) = 4 = \sqrt{16c - 16} = 4\sqrt{c-1}, \qquad c - 1 = 1, \qquad c = 2.$$

This gives the answer in Appendix 2.

26. Team Project. (b) In finding a differential equation you always have to get rid of the arbitrary constant c. For $xy = c$ this is very simple because this equation is solved for c (differentiate this equation implicitly with respect to x); in other cases it is usually best to first solve algebraically for c.

(d) This orthogonality condition is usually considered in calculus. You will need it again in Sec. 1.8.

27. CAS Project. This integral (the error function, except for a constant factor; see (35) in Appendix A3.1) is important in heat conduction (see Sec. 11.6). A similar integral is basic in statistics (see Sec. 22.8).

Sec. 1.4 Modeling: Separable Equations

Problem Set 1.4. Page 23

1. Exponential growth. Let $y(0) = y_0$ be the initial amount at $t = 0$. The model equation $y' = ky$ has the solution $y = ce^{kt}$. For the given initial amount y_0 this becomes $y = y_0 e^{kt}$. For $t = 1$ (1 day) this gives $y(1) = y_0 e^k$. By assumption this is twice the initial amount (doubling in 1 day). Hence $y_0 e^k = 2y_0$. Divide this by y_0 to get $e^k = 2$. After 3 days you have $y(3) = y_0 e^{3k} = y_0 \cdot 2^3$, where we used $e^{ab} = (e^a)^b$. Similarly for 1 week ($t = 7$).

11. Sugar inversion. $y' = ky$, $y(t) = 0.01e^{kt}$ from the first condition and $y(4) = 0.01e^{4k} = 1/300 = 0.01/3$ from the second. Hence $e^{4k} = 1/3$, $k = 1/4 \ln(1/3) = -0.275$.

15. Curves (ellipses) From calculus you know that the slope of the tangent of a curve $y = y(x)$ is the derivative $y'(x)$. From the given data you thus obtain immediately the differential equation $y' = -4x/y$. Solve it by separation of variables (multiply by y),

$$y\,dy = -4\,x\,dx, \qquad y^2/2 = -2x^2 + c, \qquad y^2/4 + x^2 = c/2.$$

For instance, $c = 2$ gives the ellipse with semi-axes 1 (in the x-direction) and 2 (in the y-direction). Sketch this ellipse and some of the others.

Sec. 1.5 Exact Differential Equations. Integrating Factors

Example 3. A nonexact equation. You can write the given equation as $y' = y/x$. Separate variables, obtaining $dy/y = dx/x$, $\ln y = \ln x + \tilde{c}$, $y = cx$.

Problem Set 1.5. Page 31

17. Test for exactness. Initial value problem. Exactness is seen from

$$\frac{\partial}{\partial y} M = \frac{\partial}{\partial y}((x+1)e^x - e^y) = -e^y,$$

$$\frac{\partial}{\partial x} N = \frac{\partial}{\partial x}(-xe^y) \qquad = -e^{-y},$$

where the minus sign in the second line results from taking the dy-term to the left in order to have the standard form of the equation. You see that the equation is exact. Integrating M with respect to x gives $u = xe^x - xe^y + k(y)$ with arbitrary $k(y)$. Differentiating this with respect to y and equating the result to N gives $-xe^y + k'(y) = -xe^y$, hence $k'(y) = 0$ and $k = const.$ This shows that a general solution is $u = xe^x - xe^y = c$. Because of the initial condition set $x = 1$ and $y = 0$, obtaining $u = e - 1$. This gives the answer $u = xe^x - xe^y = e - 1$.

23. Several integrating factors. From this problem you can learn that if an equation has an integrating factor, it has many such factors, giving essentially the same (implicit) general solution. Taking $F = y$, you obtain the equation $y^2\,dx + 2xy\,dy = 0$. To check exactness, calculate $\frac{\partial}{\partial y}(y^2) = 2y$ and $\frac{\partial}{\partial x}(2xy) = 2y$, which proves exactness. Integrating y^2 with respect to x gives $xy^2 + k(y)$. Differentiating this with respect to y and equating the result to $2xy$, you obtain for $k(y)$ the condition $2xy + k'(y) = 2xy$, $k'(y) = 0$, $k(y) = const.$ The solution is $xy^2 = const.$

Choosing $F = xy^3$ as an integrating factor gives the exact equation $xy^4\,dx + 2x^2y^3\,dy = 0$. Proceeding as before, you obtain

$$u = (1/2)x^2y^4 + k(y), \qquad 2x^2y^3 + k'(y) = 2x^2y^3, \qquad u = (1/2)x^2y^4 = C,$$

which implies $xy^2 = c$, as before.

25. Integrating factor. $P\,dx + Q\,dy = 0$ in (12) is the nonexact equation. $FP\,dx + FQ\,dy = 0$ is the exact equation obtained by multiplying with an integrating factor F. Hence $FP = M$ and $FQ = N$ play the role of M and N in an exact equation. Accordingly, the exactness condition is $\partial(FP)/\partial y = \partial(FQ)/\partial x$. In the present problem,

$$\frac{\partial}{\partial y}(FP) = \frac{\partial}{\partial y}(e^x \sin y) = e^x \cos y,$$

$$\frac{\partial}{\partial x}(FQ) = \frac{\partial}{\partial x}(e^x \cos y) = e^x \cos y,$$

which shows exactness. Integrating FP with respect to x gives $u = e^x \sin y + k(y)$. To determine $k(y)$, differentiate u with respect to y and equate the result to FQ (which now plays the role of N). This gives

$$e^x \cos y + k'(y) = e^x \cos y, \qquad k'(y) = 0, \qquad k(y) = const.$$

Hence the answer is

$$u = e^x \sin y = c = const.$$

Note that in the present case you can solve this for y; this gives

$$y = \arcsin(ce^{-x}).$$

Sec. 1.6 Linear Differential Equations. Bernoulli Equation

Example 2. The integral can be solved by integration by parts or more simply by "undetermined coefficients", that is, by setting

$$\int e^{0.05t} \cos t \, dt = e^{0.05t}(A \cos t + B \sin t)$$

and differentiating on both sides. This gives

$$e^{0.05t} \cos t = e^{0.05t}[0.05\,(A \cos t + B \sin t) - A \sin t + B \cos t].$$

Now equate the coefficients of $\sin t$ and $\cos t$ on both sides. The sine terms give $0 = 0.05B - A$, hence $A = 0.05B$. The cosine terms give

$$1 = 0.05A + B = 0.05^2B + B,$$

hence $B = 1/1.0025 = 0.997506$ and $A = 0.05B = 0.049875$. Multiplying A and B by 50 (the factor that we did not carry along) gives a and b in Example 2. The integrals in Example 3 can be handled similarly.

Problem Set 1.6. Page 38.

7. General solution. Multiplying the given equation by e^{kx}, you obtain

$$(y' + ky)e^{kx} = (ye^{kx})' = e^{kx}e^{-kx} = 1$$

and by integration, $ye^{kx} = x + c$. Division by e^{kx} gives the solution $y = (x + c)e^{-kx}$. Note that in (4) you have the integral of $e^{kx}e^{-kx} = 1$, which has the value $x + c$, so that the use of (4) is very simple, too.

17. Initial value problem. In any case the first task is to write the equation in the form (1). In the present problem,

$$y' - 2y \tanh 2x = -2 \tanh 2x.$$

In (4) you thus have $p = -2 \tanh 2x = -(\ln \cosh 2x)'$. Hence the integral h of p is $h = -\ln(\cosh 2x)$. In (4) you need $e^{-h} = \cosh 2x$ and under the integral sign $e^h = 1/(\cosh 2x)$. Since $r = -2 \tanh 2x$, the integrand is

$$-2\tanh 2x/\cosh 2x = -2\sinh 2x/(\cosh 2x)^2 = (1/\cosh 2x)'.$$

Hence the integral equals $1/(\cosh 2x) + c$. Multiplying this by $e^{-h} = \cosh 2x$ gives the general solution $y = 1 + c\cosh 2x$. From this and the initial condition, $y(0) = 1 + c = 4$, $c = 3$. *Answer*: $y = 1 + 3\cosh 2x$.

33. Bernoulli equation. This is a Bernoulli equation with $a = 4$. Hence you have to set $u = 1/y^3$. By differentiation (chain rule!) $u' = -3y^{-4}y'$. This suggests multiplying the given equation by $-3y^{-4}$, obtaining

$$-3y^{-4}y' - y^{-3} = -1 + 2x.$$

The first term is u' and the second is $-u$; thus $u' - u = 2x - 1$. Formula (4) with u instead of y gives the general solution $u = ce^x - 2x - 1$. Hence the answer is

$$y = u^{-1/3} = (ce^x - 2x - 1)^{-1/3}.$$

Sec. 1.7 Modeling: Electric Circuits

Example 1 Step 5. For the idea of evaluating the integral by undetermined coefficients, see this Manual, Sec. 1.6.

Problem Set 1.7. Page 47

7. Choice of L. This is a problem on the exponential approach to the limit, as it also occurs in various other applications. For constant $E = E_0$ the model of the circuit is $I' + (R/L)I = E_0/L$. The initial condition is $I(0) = 0$ since the current is supposed to start from zero. The general solution and the particular solution are

$$I = ce^{-Rt/L} + \frac{E_0}{R}, \qquad I = \frac{E_0}{R}(1 - e^{-Rt/L}).$$

25% of the final value of I is reached if the exponential term has the value 0.75, that is, $\exp(-Rt/L) = 0.75$. With $R = 1000$, $t = 1/10000$ by taking logarithms you obtain $0.1/L = \ln(1/0.75) = 0.2877$, so that $L = 0.1/0.2877 = 0.3476$.

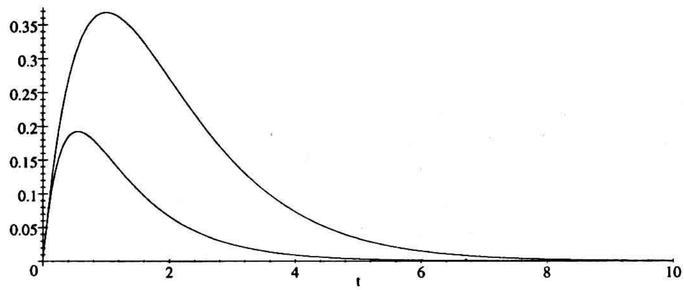
9. RL-circuit. The two cases can first be handled jointly; the difference will appear in evaluating the integral. The model is $I' + RI/L = e^{-t}/L$. You can solve it by (4) in Sec. 1.6. Since $p = R/L$, integration gives $h = Rt/L$. Hence $e^{-h} = e^{-Rt/L}$ and $e^h = e^{Rt/L}$. This yields the integrand $(1/L)\exp(Rt/L)\exp(-t) = (1/L)\exp[(R/L - 1)t]$. If $R/L - 1 = 0$, the integrand is $1/L$, and the integral is $t/L + c$. This is Case (b), the solution being

$$I = (t/L + c)e^{-t}.$$

If $R/L - 1$ is not zero, you have to integrate an exponential function, obtaining $\exp[(R/L - 1)t]/(R - L)$. This is Case (a), the solution being

$$I = \frac{e^{-t}}{R - L} + ce^{-Rt/L},$$

where the first term became simple because $\exp(-h)\exp h = 1$. The figure shows the two solutions for $I_0 = 0$, $L = 1$ and (a) $R = 3$, (b) $R = 1$. Find out which curve corresponds to (a) and which to (b). Sketch the solutions when $L = 1$, $R = 3$, and $I_0 = 1$, and compare.



Section 1.7. Problem 9. Solutions in both cases

19. Periodic electromotive forces are particularly important in practice. The simplest way of obtaining steady-state solutions is by substituting an expression of the form of the electromotive force with undetermined coefficients and determining the latter by equating corresponding coefficients on both sides of the equation. In the problem, the model equation, divided by a common factor 25, is

$$2Q' + Q = 4\cos 2t + \sin 2t + 8\cos 4t + \sin 4t.$$

The right side suggests setting

$$Q = a\cos 2t + b\sin 2t + c\cos 4t + k\sin 4t$$

By differentiation and multiplication by 2,

$$2Q' = -4a\sin 2t + 4b\cos 2t - 8c\sin 4t + 8k\cos 4t.$$

Hence you must have $a + 4b = 4$ (from $\cos 2t$), $-4a + b = 1$ (from $\sin 2t$). The solution is $a = 0$, $b = 1$. Similarly, $c + 8k = 8$ (from $\cos 4t$), $-8c + k = 1$ (from $\sin 4t$). The solution is $c = 0$, $k = 1$. Hence there are no cosine terms. The answer is $Q = \sin 2t + \sin 4t$. This "method of undetermined coefficients" will be very important in connection with vibrations in the next chapter.

Sec. 1.8 Orthogonal Trajectories of Curves. *Optional*

Problem Set 1.8. Page 51

3. Family of curves. $\cosh(x - c)$ is a translate of $\cosh x$ through the distance c to the right ($x - c = 0$ or $x = c$ corresponds to the lowest point of the curve, which is now at $x = c$, $y = 1$). Adding $-c$ moves the translated curve down. Thus, $y = \cosh(x - c) - c$. If $x = c$, then $y = -c + 1$; this is the lowest point of the corresponding curve. Make a sketch.

9. Differential equation of a family of curves. The differential equation to be derived must not contain c. This is quite essential. You accomplish this as follows. Solve the given equation algebraically for c^2,

$$c^2(x^2 - 1) + y^2 = 0, \qquad -c^2 = y^2/(x^2 - 1).$$

Differentiation with respect to x gives (chain rule!)

$$0 = \frac{2yy'}{x^2 - 1} - \frac{y^2}{(x^2 - 1)^2}\cdot 2x.$$

Dividing by $2y$ and solving algebraically for y' yields the answer shown in Appendix 2 of the text.

21. Orthogonal trajectories derive their importance from applications in electrostatics, fluid flow, heat flow, and so on. The given curves $xy = c$ are the familiar hyperbolas with the coordinate axes as asymptotes (the solid curves in Fig. 30 of the text). Differentiation with respect to x gives their differential equation

$y + xy' = 0$ or $y' = -y/x$. Formula (2) in Sec. 1.8 gives the differential equation of the trajectories $y' = +x/y$ or $yy' = x$. By integration on both sides you obtain $y^2/2 = x^2/2 + C$ or $x^2 - y^2 = c^*$, the dashed hyperbolas in Fig. 30, whose asymptotes are $y = x$ and $y = -x$ (the latter in the quadrants not shown in the figure).

Sec. 1.9 Existence and Uniqueness of Solutions. Picard Iteration

Problem Set 1.9. Page 58

1. No solution. Obtain the general solution by separating variables.

3. Vertical strip. α is the smaller of the numbers a and b/K. Since K is constant and you can now choose b as large as you please (there is no restriction in the y-direction), the smaller number is a, as claimed.

7. Linear differential equation. $y' = f(x, y) = r - p(x)\,y$ shows that the continuity of r and p makes both f and $\partial f/\partial y = -p(x)$ continuous.

11. Picard iteration. Proof by induction. You have to show that $y_n = 1 + x + \dots + x^n/n!$. This is true for $n = 0$ because $y_0 = y(0) = 1$; see (6) in Sec. 1.9. Since $y' = f(x,y) = y$, the integrand in (6) is $y_{n-1}(t)$. Make the induction hypothesis that this equals $1 + t + \dots + t^{n-1}/(n-1)!$ According to (6) you have to integrate this expression from 0 to x, obtaining $x + x^2/2 + \dots + x^n/n!$ (because $(n-1)!n = n!$), and to add $y_0 = 1$. This gives y_n, the next partial sum of the Maclaurin series of e^x, and completes the proof.

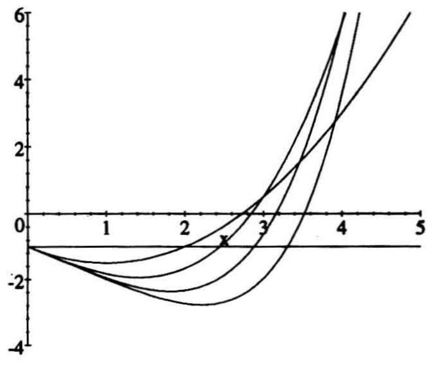
13. Picard iteration. $y' = x + y$, $y_0 = -1$.

$$y_n = -1 + \int_0^x (t + y_{n-1}(t))\,dt = -1 + \int_0^x y_{n-1}(t)\,dt + \frac{x^2}{2},$$

thus

$$y_1 = -1 - x + \frac{x^2}{2},$$

$$y_2 = -1 - x - \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^2}{2} = -1 - x + \frac{x^3}{6}, \quad \text{etc.}$$



Section 1.9. Problem 13. Picard approximations of the solution $y = -1 - x$

CHAPTER 2. Linear Differential Equations of Second and Higher Order

Sec. 2.1 Homogeneous Linear Equations of Second Order

Problem Set 2.1. Page 71

7. Reduction to first order. $y'' + e^y y'^3 = 0$ is of the form $F(y, y', y'') = 0$, so that you can set $z = y'$ and $y'' = (dz/dy)\,z$ (see Prob. 2). Substitution of this and division by z gives $dz/dy + e^y z^2 = 0$. By separation of variables, $dz/z^2 = -e^y\,dy$. Integration on both sides and multiplication by -1 gives $1/z = e^y + c_1$. Now by calculus, $z = dy/dx$ implies $dx/dy = 1/z$. Hence you can separate again and then integrate,

$$dx = (e^y + c_1)\,dy$$

$$x = e^y + c_1 y + c_2.$$

13. Motion. Expressing the given data in formulas gives $y'y'' = 1$, $y(0) = 2$, $y'(0) = 2$. By integration, $y'^2/2 = t + C$, hence $y' = \sqrt{2t + c_1}$, where $c_1 = 2C$. If you wish, you can now use the second initial condition to get $y'(0) = \sqrt{c_1} = 2$, hence $c_1 = 4$, so that $y' = \sqrt{2t+4}$. By another integration and the use of the first initial condition you obtain

$$y = \frac{1}{3}(2t+4)^{3/2} + c_2, \quad y(0) = \frac{1}{3}4^{3/2} + c_2 = \frac{8}{3} + c_2 = 2, \quad c_2 = -\frac{2}{3}.$$

This gives the answer

$$y = \frac{1}{3}(2t+4)^{3/2} - \frac{2}{3}.$$

Sec. 2.2 Second-Order Homogeneous Equations with Constant Coefficients

Problem Set 2.2. Page 75

7. General solution. Problems 1-9 amount to solving a quadratic equation (3), the characteristic equation. Observe that the solutions (4) refer to the case that y'' has the coefficient 1. For the present equation you can write $y'' - (30/9)\,y' + (25/9)\,y = 0$. Then the radicand in (4) is $225/81 - 25/9 = 0$, so that you have a double root $15/9 = 5/3$. The corresponding general solution is $y = (c_1 + c_2 x)\exp(5x/3)$.

15. Initial value problem. To solve an initial value problem, first determine a general solution by solving the characteristic equation $\lambda^2 + 2.2\lambda + 1.17 = 0$. The roots (4) are -1.3 and -0.9. The corresponding general solution is

$$y = c_1 e^{-1.3x} + c_2 e^{-0.9x}. \tag{a}$$

Because of the second initial condition you also need the derivative

$$y' = -1.3c_1 e^{-1.3x} - 0.9c_2 e^{-0.9x}. \tag{b}$$

In (a) and (b) you now put $x = 0$ and equate the result to 2 and -2.6, respectively (the given initial values), that is,

$$c_1 + c_2 = 2, \quad -1.3c_1 - 0.9c_2 = -2.6.$$

The solution is $c_1 = 2$, $c_2 = 0$, so that you get the answer $y = 2e^{-1.3x}$. Note that, in general, both solutions of a basis of solutions would appear; in that sense our present initial conditions are special.

21. Linear independence and dependence. This problem is typical of cases where one must use functional relations to prove linear dependence. Namely, $\ln x$ and $\ln(x^4) = 4\ln x$ are linearly dependent on any interval of the positive semi-axis. Graphs may help when the functions are very complicated and transformations are not so obvious as in this problem; then you may find out whether the curves of the functions look "proportional".

Sec. 2.3 Case of Complex Roots. Complex Exponential Function

Problem Set 2.3. Page 80

5. **General solution.** $y'' + 1.6\,y' + 0.64\,y = 0$ (the given equation divided by 2.5) has the characteristic equation $\lambda^2 + 1.6\lambda + 0.64 = (\lambda + 0.8)^2 = 0$ with the double root -0.8. This is Case II, with the general solution as given in Appendix 2.

7. **General solution.** Division by 16 gives $y'' - 0.5y' + 0.3125y = 0$. From (3) you thus obtain the roots

$$\lambda_1 = 0.25 + 0.5\sqrt{0.25 - 1.25} = 0.25 + 0.5i \quad \text{and} \quad \lambda_2 = 0.25 - 0.5i.$$

Note that if an equation (with real coefficients) has a complex root, the conjugate of the root must also be a root. The real part is 0.25 and gives the exponential function exp $(0.25x)$. The imaginary parts are 0.5 and -0.5 and give the cosine and sine terms. Together,

$$y = e^{0.25x}(A\cos 0.5x + B\sin 0.5x),$$

which is oscillating with an increasing maximum amplitude.

21. **Boundary value problems** will be less important to us than initial value problems. The determination of a particular solution by using given boundary conditions is similar to that for an initial value problem. In the present problem the characteristic equation is $\lambda^2 + 2\lambda + 2 = 0$. Its roots are

$$\lambda_1 = -1 + \sqrt{1-2} = -1 + i \quad \text{and} \quad \lambda_2 = -1 - i.$$

This gives the real general solution

$$y = e^{-x}(A\cos x + B\sin x).$$

On the left boundary, $y(0) = A = 1$. On the right boundary, $y(\pi/2) = B\exp(-\pi/2) = 0$, hence $B = 0$. Hence the answer is $y = e^{-x}\cos x$.

Sec. 2.4 Differential Operators. *Optional*

Problem Set 2.4. Page 83

3. **Differential operators.** $(D-2)(D+1)e^{2x} = 0$ because

$$(D-2)e^{2x} = 2e^{2x} - 2e^{2x} = 0.$$

For the second of the four given functions you first have

$$(D-2)xe^{2x} = e^{2x} + 2xe^{2x} - 2xe^{2x} = e^{2x}$$

and then

$$(D+1)e^{2x} = 2e^{2x} + e^{2x} = 3e^{2x}.$$

Similarly for the other functions.

13. **General solution.** The optional Sec. 2.4 introduces to the operator notation and shows how it can be applied to linear differential equations with constant coefficients. The facts considered are essentially as before, merely the notation changes. The given equation, divided by 10, is

$$(D^2 + 1.2D + 0.36)y = (D + 0.6)^2 y = 0.$$

It shows that the characteristic equation has the double root -0.6, so that the corresponding general solution is

$$y = (c_1 + c_2 x)e^{-0.6x}.$$

Sec. 2.5 Modeling: Free Oscillations (Mass-Spring Systems)

Problem Set 2.5. Page 90

1. **Harmonic oscillations.** Formula (4*) gives a better impression than a sum of cosine and sine terms because the maximum amplitude C and phase shift δ readily characterize the harmonic oscillation. The result follows by direct calculation, starting from the general solution

$$y = A \cos \omega_0 t + B \sin \omega_0 t$$

and using the initial conditions, first $y(0) = A = y_0$ and then

$$y' = \text{a sine term} + \omega_0 B \cos \omega_0 t, \qquad y'(0) = \omega_0 B = v_0,$$

where v suggests 'velocity'. This gives the particular solution

$$y = y_0 \cos \omega_0 t + (v_0/\omega_0) \sin \omega_0 t.$$

Accordingly, in (4*),

$$C = \sqrt{y_0^2 + \left(\frac{v_0}{\omega_0}\right)^2}, \qquad \tan \delta = \frac{v_0/\omega_0}{y_0}.$$

The derivation of (4*) suggested in the text begins with

$$\begin{aligned} y(t) &= C \cos (\omega_0 t - \delta) = C(\cos \omega_0 t \cos \delta + \sin \omega_0 t \sin \delta) \\ &= C \cos \delta \cos \omega_0 t + C \sin \delta \sin \omega_0 t = A \cos \omega_0 t + B \sin \omega_0 t. \end{aligned}$$

By comparing you see that

$$A^2 + B^2 = C^2 \cos^2 \delta + C^2 \sin^2 \delta = C^2$$

and

$$\tan \delta = \frac{\sin \delta}{\cos \delta} = \frac{C \sin \delta}{C \cos \delta} = \frac{B}{A}.$$

7. **Determination of frequencies.** $\omega_0 = \sqrt{k/m}$; see (4). Hence the frequencies are

$$\frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k_1}{m}} \quad \text{and} \quad \frac{1}{2\pi}\sqrt{\frac{k_2}{m}},$$

respectively. To prove $k = k_1 + k_2$, fix $s = s_0$ (for instance, $s_0 = 1$), choose $W_1 = k_1 s_0$ and $W_2 = k_2 s_0$, and add (couple the two systems), where k is the spring constant of the two systems

$$W = W_1 + W_2 = (k_1 + k_2)s_0 = ks_0$$

combined.

15. **Underdamping.** Equate the derivative to zero.

Sec. 2.6 Euler-Cauchy Equation

Problem Set 2.6. Page 96

3. **General solution.** Problems 2-13 are solved as explained in the text by determining the roots of the auxiliary equation (3). This is similar to the method for constant-coefficient equations in Secs. 2.2 and 2.3, but note well that the linear term in (3) is $(a - 1)\,m$, not am. Thus in Prob. 3 you have

$$m(m-1) - 20 = m^2 - m - 20 = 0.$$

The roots are -4 and 5. Hence a general solution is $y = c_1 x^{-4} + c_2 x^5$. The value $x = 0$ is excluded.

Similarly, the case of a double root of (3) gives a logarithmic term [see (7) in Sec. 2.6] and $x = 0$ and all negative x must be excluded.

7. **Pure imaginary roots**. The auxiliary equation is $m^2 + 1 = 0$. It has the roots $i = \sqrt{-1}$ and $-i$. Hence in (8) of Sec, 2.6 you have $\mu = 0$ (the real part of the roots is zero) and $\nu = 1$, so that (8) becomes simply $y = A\cos(\ln x) + B\sin(\ln x)$.

15. **Initial value problems** for Euler-Cauchy equations are solved as for constant-coefficient equations by first determining a general solution. The initial values must not be given at 0, where the coefficients of (1), written in standard form

$$y'' + \frac{a}{x}y' + \frac{b}{x^2}y = 0,$$

become infinite, but must refer to some other point, for instance, to $x = 1$. In Prob. 15 the auxiliary equation is

$$4m(m-1) + 24m + 25 = 0 \qquad \text{or} \qquad m^2 + 5m + 6.25 = 0.$$

It has the double root -2.5. The corresponding general solution (7), Sec. 2.6, is

$$y = (c_1 + c_2 \ln x)x^{-2.5}.$$

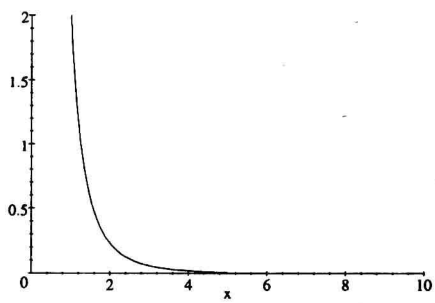
The first initial condition gives $y(1) = c_1 = 2$. For the second initial condition $y'(1) = -6$ you need the derivative. With $c_1 = 2$ the latter is

$$y' = \frac{c_2}{x}x^{-2.5} - 2.5(2 + c_2 \ln x)x^{-3.5}.$$

Setting $x = 1$, you thus obtain (since $\ln 1 = 0$)

$$y'(1) = c_2 - 5 = -6, \qquad \text{hence} \quad c_2 = -1.$$

The figure shows the particular solution obtained, $y = (2 - \ln x)x^{-2.5}$. For $x > 7.4$ the logarithm is greater than 2, so that for these x the solution becomes negative, but this can hardly be seen from the figure because the x-factor is very small in absolute value when x is large.



Section 2.6. Problem 15. Particular solution satisfying $y(1) = 2, y'(1) = -6$

Sec. 2.7 Existence and Uniqueness Theory. Wronskian

The **Wronskian** $W(y_1, y_2)$ of two solutions y_1 and y_2 of a differential equation is defined by (5), Sec. 2.7. It is conveniently written as a second-order determinant (but this is not essential for using it; you need not be familiar with determinants here). It serves for checking linear independence or dependence, which is important in obtaining bases of solutions. The latter are needed, for instance, in connection with initial value problems, where a single solution will generally not be sufficient for satisfying two given initial conditions. Of course, two functions are linearly independent if and only if their quotient is not constant. To check this, you would not need Wronskians, but we