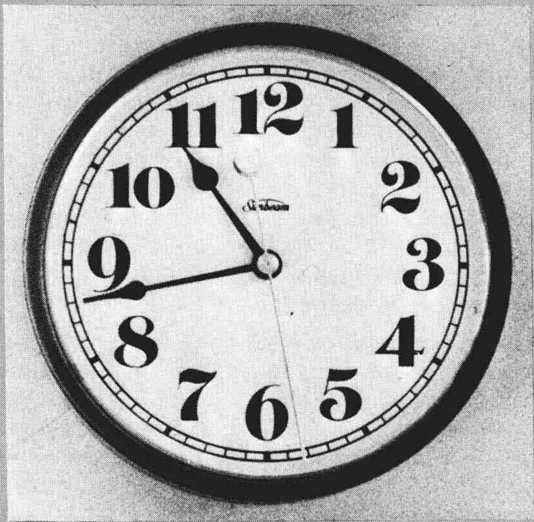
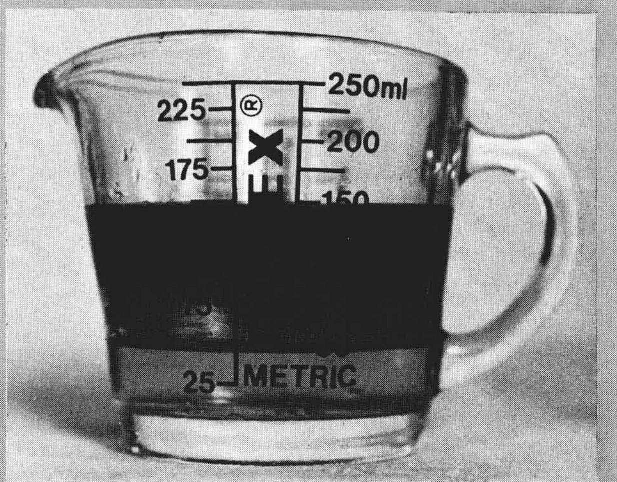
Contents



1 Mathematics is . . .

a Symbolic Language



Which is larger, 3 or 4? Did you know that one-half of 8 is o? Did you know that one-half of 8 is also 3? Questions of this type are sometimes used by elementary school teachers to introduce the distinction between a "number" and a "numeral." As you may know, "number" refers to the idea or concept, but "numeral" refers to the *symbol* used to express the idea. For example, the concept of the number three can be numerically expressed as 3, III, $4-1$, $6 \div 2$, and in many other ways. In fact, later in this chapter the symbols |||, . . . , 10_{three}, and 11_{two} will also be used to express "threeness."

This distinction between an idea or concept and the symbols used to represent it is a part of the study of mathematics at all levels. It is probably safe to say that many people have developed a dislike for mathematics because they have become too engrossed with (and confused by) the symbolism and have lost track of the ideas. You should keep in mind that symbols are supposed to be used to facilitate communication, not to cloud it. The ideas and concepts are the big issues.

We have chosen this chapter on numeration systems as our way of emphasizing the symbolic nature of mathematics. The number-numeral distinction should be kept in mind throughout the chapter. You will be adding, subtracting, multiplying, and dividing *numbers,* but you will be expressing the number facts using a variety of *numerals.*

The Hindu-Arabic numeration system was named after the Hindus, who have been given credit for inventing it, and after the Arabs, who were responsible for transmitting it to Western Europe.

1.1 Numeration Systems

Before looking at some ancient numeration systems, we want to emphasize four features of the Hindu-Arabic system we use every day. These features can then be used as points of comparison as other systems are presented.

First, the Hindu-Arabic system uses only *ten symbols,* called *digits*—0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. With the use of some or all of these digits every counting number, no matter how small or how large, can be represented.

Second, the Hindu-Arabic system does have a *symbol for "zero."* The real significance of this fact will be emphasized in a moment when the idea of place value is discussed. It will also be interesting to note a bit later that some of the ancient numeration systems did not have a symbol for "zero" and, in fact, did not really need one.

Third, the Hindu-Arabic system is a *positional system.* This means that the position of each digit within a numeral is meaningful. For example, 13 has a different meaning from 31.

Finally, the feature of the Hindu-Arabic system that ties everything together so neatly is the idea of *place value.*

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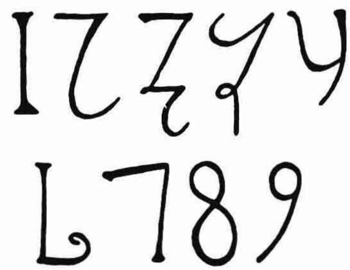
More specifically, the system uses the concept of place value and a base of ten. This is a very powerful idea that we may take for granted because we have used it so long, but let's think for a minute about what it really means. When we write 23 we mean two piles of ten and three ones, but 32 means three piles of ten and two ones. Furthermore, 479 means four piles of ten tens and seven piles of ten and nine ones. Now the fact that we have a symbol for zero becomes very significant. For example, 502 means five piles of ten tens and zero piles of ten and two ones. (Note how all four features of the Hindu-Arabic system that we have discussed are being used.)



Above is the oldest example of our numerals known in any European manuscript. This manuscript was written in Spain in 976.

Now let's take a look at a few ancient numeration systems with respect to the features discussed relative to the Hindu-Arabic system. The Egyptian hieroglyphic system is one of the oldest systems for which historical records are available; it dates back to somewhere around 3000 B.C. It was a very simple but inefficient nonpositional system. There was no place value, but the idea of simply grouping in powers of ten was used. The following chart shows some of the symbols used and the numbers they represented.

Hindu-Arabic numeral	*Egyptian numeral*	*Object symbol represented*
1	|	vertical staff
10	∩	heel bone
10^2	𓍢	scroll
10^3	𓆼	lotus flower
10^4	𓂭	pointing finger
10^5	𓆐	burbot fish
10^6	𓁨	astonished man

The symbols were carved on wood or stone. The numeral for a particular number could then be "carved" by using the appropriate symbols in an additive fashion. For example, thirty-five would be thought of as three tens and five ones and thus would be expressed as ∩∩∩||||| or ∩∩∩||||| or |||||∩∩∩. (The Egyptians probably used something like the last numeral since it was customary to write from right to left.) Following are a few more examples of Egyptian numerals.

Hindu-Arabic numeral	*Egyptian numeral*
123	𓍢∩∩|||
2121	𓆼𓆼𓍢∩∩|
11167	𓂭𓆼𓍢∩∩∩∩∩∩|||||||
104	𓍢||||

The word *sunya,* which meant "empty" or "blank," was used by the early Hindus to mean zero. This was translated into the Arabic term *sifr* which in Latin became *zephirum* or *zephyrum.* Other Latin forms such as *zeuro, zepiro, cifra,* and *cifre* were also used. From these we can see the origins of our words "zero" and "cipher."

The Egyptian system did not have a symbol for zero and, as the last example illustrates, did not really need one.

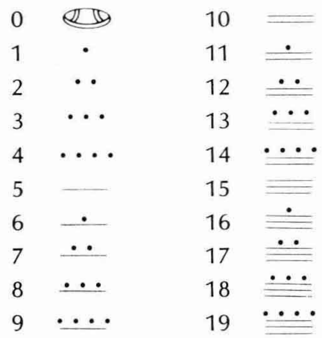
Next let's take a brief look at the Roman numeration system, which is very similar to the Egyptian system. (Most of this should look rather familiar since we still use the Roman numerals on clocks, for page numbering, and other applications.) The Roman system also uses the idea of grouping in powers of ten, but they tossed in a few extra symbols. The common Roman symbols used are as follows:

Hindu-Arabic	1	10	10^2	10^3	5	50	500
Roman	I	X	C	M	V	L	D

The system in its early stages worked exactly like the Egyptian system except for the symbols used. Thus, the Roman numeral MMCCCXII meant $2000 + 300 + 10 + 2 = 2312$. Likewise 154 would appear as CLIIII. The order of the symbols was not important since no concept of position was used. However, as the system developed, a kind of positional concept usually referred to as the *subtractive principle* was introduced. It simply stated that when a symbol for a larger number is preceded by a symbol for a smaller number, then subtraction is indicated. Thus, IV means 5 subtract 1, or 4. Likewise, XL means 50 subtract 10, or 40. The only problem is that the principle is not used consistently. For example, 49 is not written as IL, but as XLIX. The subtractive principle applies only to the numerals IV, IX, XL, XC, CD, and CM.

Another later development of the Roman numeration system was the use of a line above a symbol to represent "1000 times as great." Thus, $\overline{L}$ means 50,000, $\overline{XL}$ means 40,000, and $\overline{CCLII}$ means 252,000. Perhaps before leaving the Roman system it would be interesting for you to check our entries in the following chart.

Hindu-Arabic numeral	*Roman numeral*
78	LXXVIII
104	CIV
427	CDXXVII
94	XCIV
5179	$\overline{V}$CLXXIX
92,384	$\overline{XCII}$CCCLXXXIV



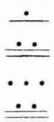
The Mayan Indians of Central America developed a very interesting numeration system. The exact date of its origin is unknown, but evidences of it were found by Spanish explorers in the 16th century. It is a positional system with a type of place value, but written in a vertical format. The system is "primarily" a base twenty system and does have a symbol for zero. The twenty basic symbols are shown in the margin.

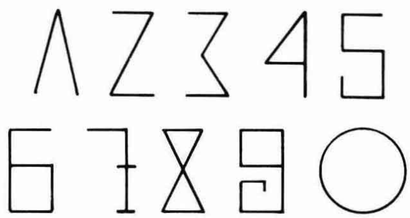
As previously mentioned, the numerals were written in a vertical format and from bottom to top represented place values of 1, 20, (18)(20), $(18)(20)^2$, $(18)(20)^3$, $18(20)^4$, etc. So you see that they did have a type of base twenty system except that beginning at the third place from the bottom a factor of 18 was used instead of another factor of 20. (Historians tend to agree that this might have been due to the fact that the Mayan year consisted of 360 days.) Thus, the numeral

$3(18)(20)^2$
$0(18)(20)$
$12(20)$
6

represents $3(18)(20)^2 + 0(18)(20) + 12(20) + 6 = 21600 + 0 + 240 + 6 = 21846$. (Notice the breakdown above.) Note that spacing between successive place value positions is necessary to avoid confusion.

See if you agree with the following Mayan representations:

Mayan numeral	*Hindu-Arabic numeral*
	$2(20)+17=57$
	$1(18)(20)+2(20)+0=400$
	$1(18)(20)^2+1(18)(20)+14(20)+5 = 7845$
	Now it's your turn to write down what it means. $=3672$
	Try again. $=47597$

The first set of problems is designed to give you the opportunity to play around a little with these numeration systems. Hopefully, as you do so, the power and convenience of the Hindu-Arabic system will become even more significant to you. Don't forget that the main issue in this chapter is the idea that "mathematics is a symbolic language."



An interesting (and logical) way in which our Arabic numerals might have been conceived is as follows. As shown here, the numerals were shaped so that each contains the appropriate number of angles. Thus the numeral for "one" contains one angle, "two" has two angles, and so on. (From Eves, *In Mathematical Circles Revisited,* p. 3.)

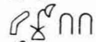
Problem Set 1.1

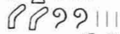
1. Express each of the following as a Hindu-Arabic numeral:

(a) 

(b) 

(c) 

(d) 

(e) 

(f) 

2. Express each of the following as an Egyptian numeral:

(a) 38 (b) 125 (c) 2214 (d) 4027
(e) 2403 (f) 20001 (g) 100003 (h) 2100014

3. Explain why a symbol for the number zero was not really needed in the Egyptian system.

4. Explain why ||∩ or ∩|| or |∩| all represent the same number in the Egyptian system. What number do they represent?

5. Express each of the following as a Hindu-Arabic numeral:

(a) XIV (b) LXIV (c) CXXXII
(d) MCCXVI (e) CDXVII (f) CMXLIV
(g) $\overline{\text{X}}$CLIV (h) $\overline{\text{CXXI}}$XLII

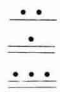
6. Express each of the following as a Roman numeral:

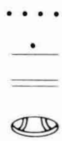
(a) 27 (b) 93 (c) 134 (d) 391
(e) 483 (f) 1456 (g) 1977 (h) 32,413
(i) 81,001 (j) 156,024

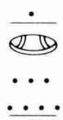
7. Express each of the following as a Hindu-Arabic numeral:

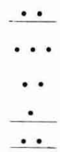
(a) 

(b) 

(c) 

(d) 

(e) 

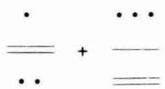
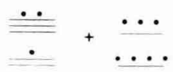
(f) 

8. Express each of the following as a Mayan numeral. (Think carefully and don't forget what each "position" of their place value represents.)

(a) 35 (b) 176 (c) 359 (d) 360
(e) 417 (f) 1804 (g) 134912

(Give this your best shot *before* you look up the answer in the back.)

9. Do the following additions, expressing the answer in the same system as used in the problem. Don't forget to "regroup" when it is appropriate to do so.

(a) ∩|||||| + ∩∩||||||| (b) ∩∩∩∩∩∩|| + ∩∩∩∩∩|||
(c) XIV + CIX (d) CCXXXII + CXIII
(e)  (f) 

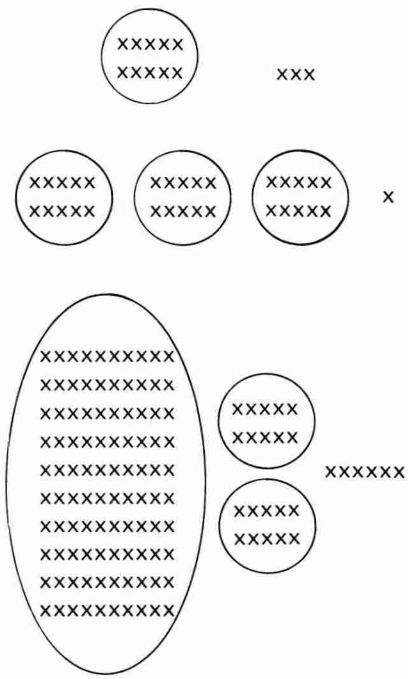
Which of the three systems (Egyptian, Roman, Mayan) is the most convenient for manipulation purposes? Why?

10. **Project:** Give a report on each of the following numeration systems. Be sure to include a discussion of (1) basic symbols; (2) concept of position; and (3) concept of place value. Encyclopedias and books on the history of mathematics would be good sources.

(a) Babylonian (b) Greek (c) Chinese-Japanese

1.2 Place Value Systems with Bases Other than Ten

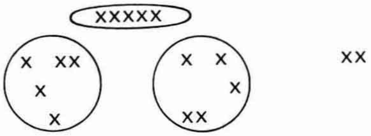
As mentioned in Section 1.1, the Hindu-Arabic numeration system is a place value system using a base of ten. Let's take a brief look at the meaning of that statement before investigating systems with bases other than ten. The fact that a base of ten is used simply means that we group in sets of ten. For example, consider the first set of xs in the margin, grouped as indicated. We say that there is one group of ten xs and three single xs and write 13. The "1" represents the one group of ten and the "3" represents the three. Now consider a second set of xs grouped as indicated. This time there are three groups of ten xs and one single x, and the numeral 31 represents this number idea. Consider a third set of xs grouped as indicated. The numeral 126 is used to symbolize the one group of ten tens, the two groups of ten, and the six ones. Thus, 126 means $1(10)^2 + 2(10) + 6$ and this latter form of the numeral is called the **expanded form**. As we see, the expanded form clearly emphasizes the place value concept. The following examples illustrate this same idea.



$$4768 = 4(10)^3 + 7(10)^2 + 6(10) + 8$$
$$97423 = 9(10)^4 + 7(10)^3 + 4(10)^2 + 2(10) + 3$$
$$621479 = 6(10)^5 + 2(10)^4 + 1(10)^3 + 4(10)^2 + 7(10) + 9$$
$$a_n \ldots a_4a_3a_2a_1a_0 = a_n(10)^n + \cdots + a_4(10)^4 + a_3(10)^3 + a_2(10)^2 + a_1(10)^1 + a_0$$

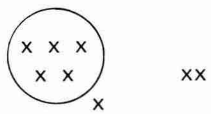
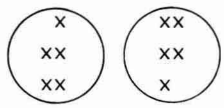
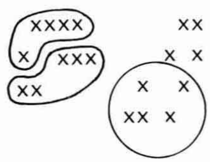
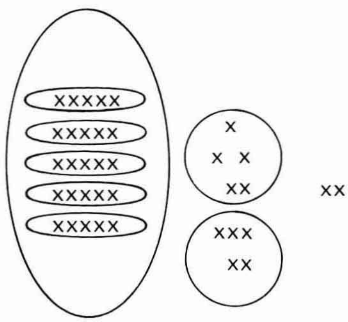
Don't let the last example bother you. It is included simply to illustrate a general representation for any counting number.

The choosing of symbols is technically very arbitrary. We could have made up our own symbols and in fact have done this in one problem in Problem Set 1.2. Usually the common Hindu-Arabic numerals are used with subscripts attached, as you will see in a moment.



Now suppose that instead of working in a base ten system, we wanted to use a **base five** system. First of all, this means that we have but five distinct symbols to use, namely 0, 1, 2, 3, and 4. From a place value standpoint it means that we are going to group in sets of five. Thus, the pile of xs in the margin would be grouped as indicated. There are three piles of five xs and two single xs and so we write 32_{five}. The "3" represents the three piles of five, the "2" represents the two single xs, and the "five" used as a subscript indicates the fact that base five is being used. (If no subscript appears, then you are to assume that it is the usual base ten symbol unless the context of the material definitely indicates otherwise.) The 32_{five} is read as "three, two, base five" not as "thirty-two base five." The word "thirty" refers to a base ten vocabulary.

The following chart indicates some additional groupings of xs using a base five system. Make sure that you understand the base five numeral representations for each of these.

Piles of xs	*Base five numeral*
	13_{five}
	20_{five}
	34_{five}
	122_{five}

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Special attention should be called to the last example. Note that we have one group of five fives, two groups of five, and two single xs.

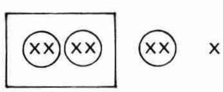
All of us are quite proficient at counting when using a base ten system. Now let's think about the counting process in a base five system. Study the following chart very carefully and be sure that you could continue the process of writing base five numerals for the counting numbers.

Sets of xs to be counted	*Base ten numeral*	*Base five numeral*
{ }	0	0_{five}
{x}	1	1_{five}
{xx}	2	2_{five}
{xxx}	3	3_{five}
{xxxx}	4	4_{five}
{xxxxx}	5	10_{five}
{xxxxxx}	6	11_{five}
{xxxxxxx}	7	12_{five}
{xxxxxxxx}	8	13_{five}
{xxxxxxxxx}	9	14_{five}
{xxxxxxxxxx}	10	20_{five}

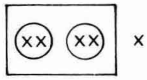
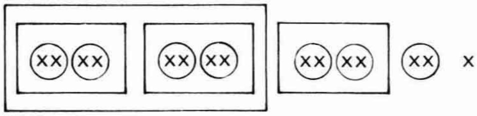
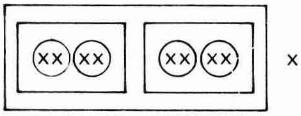
Which base five numeral would come after 24_{five}? How about after 34_{five}? How about after 44_{five}?

Next, let's do a little work in a **base two** (binary) system. This means that we have only two distinct symbols to use and we will choose the ordinary "0" and "1" symbols. It also means that we are going to group in sets of two. Thus, the pile of xs shown here would be grouped as indicated.

The numeral 11_{two} indicates the one pile of two xs and the one single x. Look over the following chart carefully and be sure that you understand the base two representations for each of the piles of xs.

Piles of xs	*Base two numeral*
	10_{two}
	111_{two}

It is true that mathematical symbolism has been somewhat standardized throughout the world; however, variations still do exist. For example, in England the decimal point is positioned halfway up (like our use of the dot for multiplication) and not on the bottom. It is also not uncommon to find inparts of continental Europe the use of a comma in place of a decimal point.

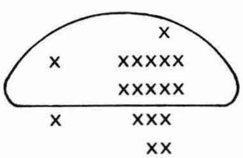
Piles of xs	*Base two numeral*
	101_{two}
	1111_{two}
	1001_{two}

As you can see, base two is simple because it has only two distinct symbols to work with, but it is a little confusing because you have to "repile" so frequently. As will be emphasized in Chapter 11, the fact that base two uses only two distinct symbols makes it very practical in today's computer world.

The "counting process" using base two symbols is started in this next chart. Again, be sure that you understand all of our entries and also that you are able to continue the process.

Set of xs to be counted	*Base ten numeral*	*Base two numeral*
{ }	0	0_{two}
{x}	1	1_{two}
{xx}	2	10_{two}
{xxx}	3	11_{two}
{xxxx}	4	100_{two}
{xxxxx}	5	101_{two}
{xxxxxx}	6	110_{two}
{xxxxxxx}	7	111_{two}
{xxxxxxxx}	8	1000_{two}
{xxxxxxxxx}	9	1001_{two}
{xxxxxxxxxx}	10	1010_{two}

Which base two numeral would come after 1111_{two}? How about after 100010_{two}? How about after 100111_{two}?



Now let's work with a base bigger than ten, such as **base twelve.** Having a base of twelve means that we need twelve distinct symbols; the ones most commonly used are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, *t*, *e*. Furthermore, base twelve means that the grouping process will be done in sets of twelve. Thus, the pile of xs shown here would be grouped as indicated and the

numeral 16_{twelve} would be used to represent the one pile of twelve xs and the six single xs. Study the following chart and be sure that you understand the base twelve representations for each of the piles of xs.

Piles of xs		*Base twelve numeral*
(xxxxxx xxx xxx)	xx	12_{twelve}
(xxxxxx xxxxxx) (xxxxxx xxxxxx)	xxxxx	25_{twelve}
(xxxxxx xxxxxx)	xxxxxxxxxx	$1t_{twelve}$
(xxxxxx xxxxxx) (xxxxxx xxxxxx)	xxxxxx xxxxx	$2e_{twelve}$

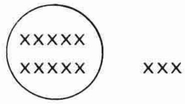
The "counting process" using base twelve symbols is started in the following chart. Study it carefully and be sure that you can continue writing base twelve numerals for the counting numbers.

Set of xs to be counted	*Base ten numeral*	*Base twelve numeral*
{ }	0	0_{twelve}
{x}	1	1_{twelve}
{xx}	2	2_{twelve}
{xxx}	3	3_{twelve}
{xxxx}	4	4_{twelve}
{xxxxx}	5	5_{twelve}
{xxxxxx}	6	6_{twelve}
{xxxxxxx}	7	7_{twelve}
{xxxxxxxx}	8	8_{twelve}
{xxxxxxxxx}	9	9_{twelve}
{xxxxxxxxxx}	10	t_{twelve}
{xxxxxxxxxxx}	11	e_{twelve}
{xxxxxxxxxxxx}	12	10_{twelve}
{xxxxxxxxxxxxx}	13	11_{twelve}

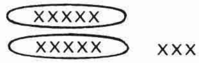
Measuring the grain and the granary. Egyptian relief courtesy of the Museum of Fine Arts, Boston.

Which base twelve numeral comes after 19_{twelve}? How about after $2e_{twelve}$? How about after ee_{twelve}?

In this section, we have made frequent use of "piling xs" to emphasize the place value concept of different bases. This same strategy can be used to change numerals from one base to another base. For example, suppose that we want to change 13_{ten} to a base five numeral. We know that 13_{ten} means "one pile of ten and three ones." Thus, using xs we have the following situation.

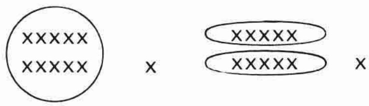


To change to base five we need to regroup in sets of five.

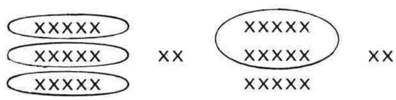


Therefore, we have "two piles of five and three single xs" and this can be written as 23_{five}. Study the following diagrams and see if you can complete the statements. In each case, the diagram should help you change the numeral on the left to the base indicated on the right.

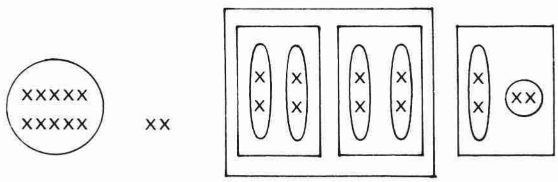
(a) $11_{ten} = \underline{\qquad}_{five}$



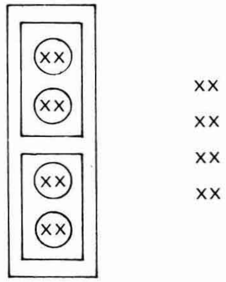
(b) $32_{five} = \underline{\qquad}_{ten}$



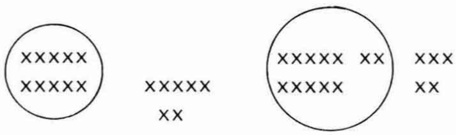
(c) $12_{ten} = \underline{\qquad}_{two}$



(d) $1000_{two} = \_\_\_\_\_\__{ten}$



(e) $17_{ten} = \_\_\_\_\_\__{twelve}$



Your answers should be (a) 21_{five}, (b) 17_{ten}, (c) 1100_{two}, (d) 8_{ten}, and (e) 15_{twelve}.

As the final point of this section, consider the following pile of xs in the margin and the various numerical representations indicated. In other words, the same number concept can be represented by many different numerical (symbolic) forms.

19_{ten}
34_{five}
10011_{two}
17_{twelve}

Mathematics is indeed a symbolic language and one needs to understand both the symbolism and the ideas being represented.

Problem Set 1.2

1. Represent the number of xs below in (a) base five, (b) base two, and (c) base twelve notation.

xxxxx
xxxxx
xxxx

2. Write the base five numerals for the counting numbers from 30_{five} to 112_{five}, inclusive.
3. Write the base two numerals for the counting numbers from 111_{two} to 100000_{two}, inclusive.
4. Write the base twelve numerals for the counting numbers from 90_{twelve} to 100_{twelve}, inclusive.

5. Write the numeral in the base indicated for the *next consecutive counting number* for each of the following.

(a) 34_{five} (b) 134_{five} (c) 1101_{two}
(d) 1010_{two} (e) 59_{twelve} (f) $40te_{twelve}$

6. Draw xs and group them to illustrate the meaning of each of the following.

(a) 23_{five} (b) 41_{five} (c) 100_{two}
(d) 110_{two} (e) 32_{twelve} (f) $1e_{twelve}$

7. Use xs and the grouping and regrouping ideas to complete each of the following.

(a) $18_{ten} = \_\_\_\_\__{five}$ (b) $23_{ten} = \_\_\_\_\__{five}$
(c) $24_{five} = \_\_\_\_\__{ten}$ (d) $21_{five} = \_\_\_\_\__{two}$
(e) $23_{ten} = \_\_\_\_\__{twelve}$ (f) $1011_{two} = \_\_\_\_\__{five}$
(g) $34_{ten} = \_\_\_\_\__{five}$ (h) $34_{ten} = \_\_\_\_\__{twelve}$

The following problems are designed to give you the opportunity to work with some bases other than two, five, and twelve.

8. Consider a *base four* numeration system using the four symbols 0, 1, 2, and 3.

(a) Use a base four numeral to represent the number of xs in the following pile.

xxxx
xxxx xx
xxxx

(b) Draw xs and group them to illustrate the meaning of each of the following: (i) 12_{four}, (ii) 23_{four}, (iii) 30_{four}, (iv) 123_{four}, (v) 102_{four}.
(c) Write the base four numerals for the counting numbers from 0_{four} to 100_{four}, inclusive.
(d) Use xs and the grouping and regrouping ideas to complete each of the following:

(i) $9_{ten} = \_\_\_\_\__{four}$ (ii) $13_{ten} = \_\_\_\_\__{four}$
(iii) $22_{four} = \_\_\_\_\__{ten}$ (iv) $112_{four} = \_\_\_\_\__{ten}$

9. Consider a *base seven* numeration system using the seven symbols 0, 1, 2, 3, 4, 5, and 6.

(a) Use a base seven numeral to represent the number of xs in the following pile.

xxxxxxx
xxxxxxx xxx

(b) Draw xs and group them to illustrate the meaning of each of the following: (i) 16_{seven}, (ii) 24_{seven}, (iii) 32_{seven}, (iv) 40_{seven}.

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