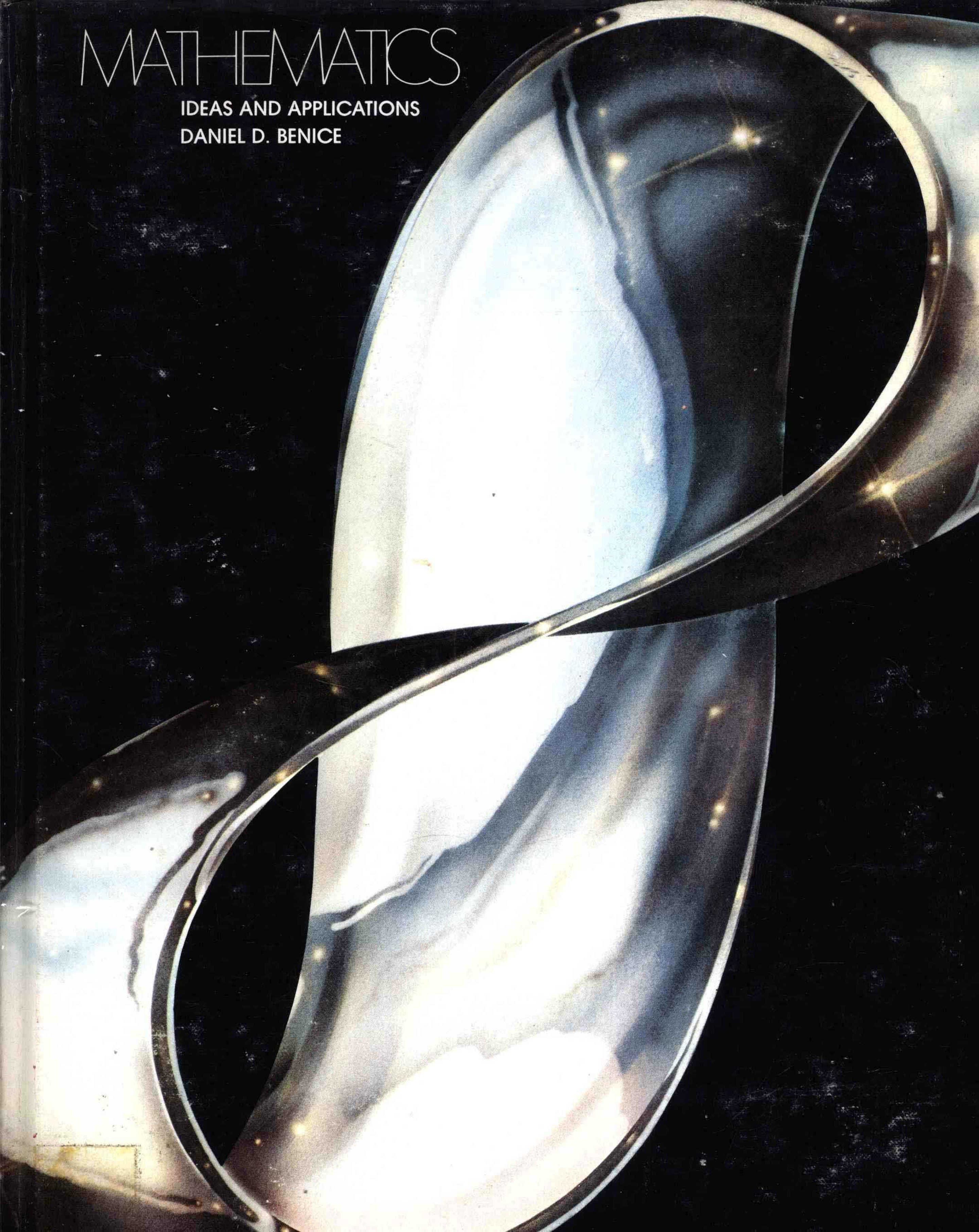
MATHEMATICS

IDEAS AND APPLICATIONS

DANIEL D. BENICE



Mathematics: Ideas and Applications

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In memory of my mother
Rose Laufer Benice

Preface

Mathematics: Ideas and Applications is designed to provide an exciting semester or two of mathematics for students in liberal arts, general education, and elementary education. The course may be of a general, survey, cultural, or appreciation nature. The book presents fresh and exciting topics, a variety of applications, historical notes where appropriate, and interesting exercises. In search of fresh ideas and applications, I have examined books and journals in many liberal arts disciplines. The result of this effort can be seen here in the application of mathematics to many fields, including: anthropology, archaeology, art, astronomy, biology, business, chemistry, computers and data processing, ecology, economics, electronics, farming, geography, geology, history, linguistics, medicine, mineralogy, music, physics, psychology, sociology, and others. Also presented are applications of mathematics to sports, consumerism, logic, games, tricks, and recreational mathematics.

Students will use arithmetic skills as they study the various applications. Although some of the applications do require a little algebra, manipulation is rarely needed.

There are no strange formulas to memorize, no busy work, and no drill. Instead, students will look at different branches of mathematics from an elementary point of view—complete with an abundance of inspiring applications. The approach is direct, moving quickly to demonstrate the usefulness of the mathematics introduced. Reviewers have used such words as "captivating" and "gemlike" to describe these applications. I expect you too will enjoy them.

Often an entire section is an application. Other times the section develops the mathematical idea and then the exercises present several applications and further development. I cannot overemphasize the importance of the exercises to the content and adaptability of the book. Since many of the exercises are in fact applications, the heading for exercises reads "Applications and Exercises."

For the most part, sections can be omitted freely, and the eight chapters can be taught in almost any order. Flexibility in topic selection and in handling of each topic chosen are built in. Although specific guidelines and outlines are presented in the *Instructor's Manual*, a note seems in order here. Chapter 1 is intended to be a stimulus and starting point for the further investigation that is presented in Chapters 2 through 7. Chapter 8 on computers can be covered at any point and then used throughout the remainder of the course. My personal preference is that Chapter 8 be used last because of its summary nature. The *Instructor's Manual* includes teaching suggestions, references to outside sources, additional exercises, and the answers to those exercises which are not answered in the text.

The book is intended to convey the importance of mathematics as well as its beauty. I expect that in the end students will have enjoyed a mathematics course, gained an appreciation for mathematics, and raised their level of understanding and ability.

I am pleased to acknowledge the assistance provided by the editorial and production staffs at Academic Press. I would also like to credit the following people who reviewed the manuscript for Academic Press and made many helpful suggestions: Bernard Eisenberg (Kingsborough Community College), Redford Fowler Yett (University of Southern Alabama), Eugene Gover (Northeastern University), James T. Hardin (formerly of Philadelphia Community College), Kenneth Retzer (Illinois State University), Donald Short (San Diego State University), and James Snow (Lane Community College).

DANIEL D. BENICE

Mathematics: Ideas and Applications

Contents

Patterns

1

"A mathematician, like a painter or poet, is a maker of patterns." These words from G. H. Hardy's *A Mathematician's Apology* may not seem consistent with the way you see mathematics. Perhaps you think of mathematics as mostly calculations and manipulations. If you do, then there is a pleasant surprise for you as you read this book. This chapter and much of the rest of the book are designed to show you the beauty and fascination of patterns in many areas of mathematics. With this chapter we hope to whet your appetite. Each of the chapters that follows presents a different branch of mathematics from both an intuitive and a practical point of view. In several instances you will see that what begins as "fun and games" often leads to important applications.

FIBONACCI SEQUENCES AND OTHER PATTERNS

Patterns are often present in *sequences* of numbers. Such sequences arise in a variety of settings, as you will soon see. Before we look at the first application, however, a bit of orientation is in order. Here are three sequences and a brief explanation or description of each.

1, 2, 3, 4, 5, . . .	The counting numbers, or natural numbers
2, 4, 6, 8, 10, . . .	Positive even numbers
1, 4, 9, 16, 25, . . .	Squares of natural numbers

The three dots after the numbers of each sequence indicate that the pattern continues indefinitely. Each number of a sequence is called a *term* of the sequence.

The following puzzle yields an interesting sequence.

> How many pairs of rabbits can be produced from a single pair of rabbits in a year, if each pair produces a new pair every month, deaths do not occur, and only rabbits that are at least one month old reproduce?

At the beginning there is 1 pair of rabbits. After one month there is still 1 pair of rabbits, but now they are able to produce a pair of rabbits. So, after two months there are 2 pairs of rabbits. One of these pairs is capable of producing and one pair is not yet able to do so. After three months there are 3 pairs of rabbits. Two of the 3 pairs are capable of producing, so at four months there are 5 pairs of rabbits. The first two terms of the sequence being generated are 1, after which each term is the sum of the two preceding terms of the sequence.

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

Courtesy of Columbia University Library

Leonardo Fibonacci
(1180–1250)

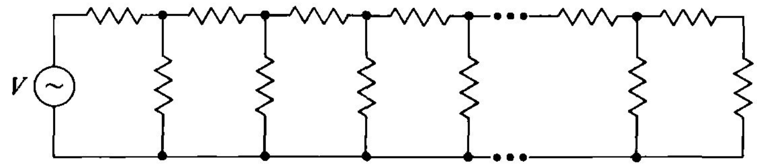
This sequence was originally presented, along with the rabbit problem, by Leonardo of Pisa in 1202 when he introduced algebra into Italy. He was also known as Leonardo Fibonacci (Leonardo, son of Bonacci) and is considered the greatest mathematician of the thirteenth century. The sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, etc. is called the *Fibonacci sequence.*

It is not difficult to see why the Fibonacci sequence contains terms that are the sum of the two preceding terms. Note, for example, that when there are 21 pairs of rabbits, 13 of the pairs have been around for a month and are therefore capable of producing more rabbits. This means that those 13 pairs will produce 13 pairs. Combine these 13 pairs of offspring with the present 21 pairs of rabbits to get the next total of 34 pairs of rabbits.

Numbers from Fibonacci's sequence occur in a variety of places in nature. A sunflower blossom has spirals that go in opposite directions. The number of spirals going in one direction and the number that go in the opposite direction are nearly always two consecutive numbers from the Fibonacci sequence.

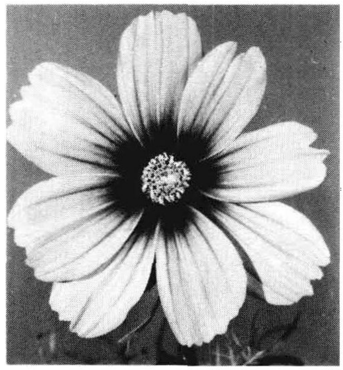
A similar relationship also exists for pineapples, evergreen cones, daisies, and some other plants. The number of petals of some flowers are numbers from the Fibonacci sequence: lily (3 petals), buttercup (5), delphinium (8), marigold (13), aster (21), daisy (34, 55, or 89).

Those of you with backgrounds in electronics may be interested to know that the numbers of the Fibonacci sequence can be generated by a specially designed electrical circuit. If all of the elements indicated by ≷ in the circuit below are 1-ohm resistors, and the current in the last branch is 1 ampere, then the voltages across the resistors are (from right to left) the Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...



Courtesy of W. Atlee Burpee Co.

Little Pinkie; 5 petals



Cosmos: 8 petals

Tithonia: 13 petals

The sequence 1, 1, 2, 3, 5, 8, 13, . . . is often considered to be *the* Fibonacci sequence because it is the original Fibonacci sequence. But any sequence in which each term after the second is the sum of the two preceding terms can be considered a Fibonacci sequence. In this sense, the following are two more examples of Fibonacci sequences.

4, 7, 11, 18, 29, 47, . . .

5, 1, 6, 7, 13, 20, 33, . . .

It does not matter what the first two terms of the sequence are. As long as each of the other terms is the sum of the two terms preceding it, the sequence is called a Fibonacci sequence.

French mathematician Edouard Anatole Lucas (1842–1891) introduced the sequence 2, 1, 3, 4, 7, 11, 18, . . . and derived some relationships among the numbers of other individual Fibonacci sequences. Consequently, Fibonacci sequences other than the original Fibonacci sequence are sometimes called Lucas sequences. Astronomers have noticed that eclipses of the sun and moon show certain patterns of repetition every 6, 41, 47, 88, 135, 223, and 358 years. Notice that these numbers form a Lucas sequence.

Here's a magic trick that uses a Fibonacci sequence. You (the magician) can quickly determine the sum of any ten consecutive Fibonacci numbers that anyone writes down in sequence. How? The sum is always 11 times the seventh number in the sequence. For the numbers 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, the sum is equal to 11 times 55, $11 \cdot 55$, which is 605. (You might want to check this by adding all the numbers.) All you need is a quick way of multiplying 11 times any number n. Here are two ways. One way: $11 \cdot n$ is the same as $10 \cdot n + 1 \cdot n$ or $10 \cdot n + n$. For example, $11 \cdot 12 = 10 \cdot 12 + 12 = 120 + 12 = 132$. Another way: To multiply 11 times any number, write the number and write the same number below it but shifted one place to the left, and add. Thus, $11 \cdot 12$ is

$$\begin{array}{r} 12 \\ 12 \\ \hline 132 \end{array}$$

which is a shortcut for

$$\begin{array}{r} 12 \\ 11 \\ \hline 12 \\ 12 \\ \hline 132 \end{array}$$

Make up a few Fibonacci sequences and verify that this trick works. In Application 4 we'll see *why* it works.

APPLICATIONS AND EXERCISES

1. Solve Fibonacci's problem: How many pairs of rabbits can be produced in a year?

2. Fill in the blanks in the following Fibonacci (or Lucas) sequences.

(a) 2, 6, 8, ____, ____, ____, ____, ____
(b) 2, 3, 5, ____, ____, ____, ____, ____, ____
(c) ____, ____, ____, ____, ____, 131, 212
(d) ____, ____, ____, ____, 36, ____, 95
(e) 35, ____, 58, ____, ____, ____

3. If a_n is any term (the nth term) of a sequence, then a_{n+1} is the term after it and a_{n-1} is the term before it.

$$\ldots, a_{n-1}, a_n, a_{n+1}, \ldots$$

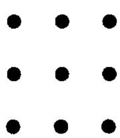
Use a_{n-1}, a_n, and a_{n+1} to express the relationship among the terms of a Fibonacci sequence.

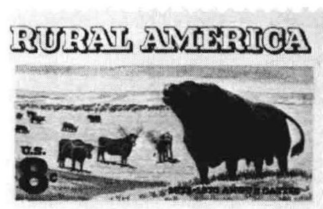
Photos by I. J. Neubart

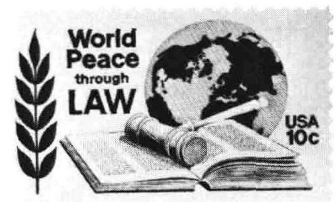
4. Now let's find out *why* the magic trick presented in this section does indeed work. Let the first of the ten terms of a Fibonacci sequence be called x, and let the second term be y. Then the third term is $x + y$, the sum of the first two terms.
   (a) Show that the fourth term is then $x + 2y$.
   (b) Show that the fifth term is $2x + 3y$.
   (c) Find the sixth through tenth terms. As a check, note that the tenth term is $21x + 34y$.
   (d) Add all ten terms. You should get $55x + 88y$ as the total.

Now look at the seventh term, which is $5x + 8y$. What is the relationship between the seventh term, $5x + 8y$, and the sum, $55x + 88y$? The sum is 11 times $5x + 8y$. In other words, the sum is 11 times the seventh term of the sequence.

5. One of the first sequences we looked at in this section was a sequence consisting of the squares of numbers. Do you know why 3^2, or $3 \cdot 3$, is called 3 *squared?* Examine the illustration below and write a brief explanation.







6. The cost of sending a letter changed from 3¢ in 1957 to 13¢ in 1976. The sequence of costs: 3¢, 4¢, 5¢, 6¢, 8¢, 10¢, 13¢.
   (a) Show that the increase from 4¢ to 5¢ represents a 25% increase in cost.
   (b) Which cost change represents the smallest percentage increase?
   (c) Which cost change represents the largest percentage increase?

Pascal's Triangle

The array of numbers shown below is not a sequence, but it is a pattern with applications in the study of probability (see page 340). It is called *Pascal's triangle,* after the French mathematician Blaise Pascal, who introduced it to the Western world in 1665. The triangle had appeared more than 200 years earlier, however, in publications by the Chinese mathematicians Yan Yui and Chu shih-chieh.

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

7. (a) The triangle shown contains only five rows. There is a sixth row, a seventh row, and so on. The numbers in each row are determined by the numbers in the row above it. Study the triangle to determine the pattern used to obtain a row from the row that precedes it. (*Hint:* An arithmetic operation is used.)

(b) Obtain row six.

(c) Obtain row seven.

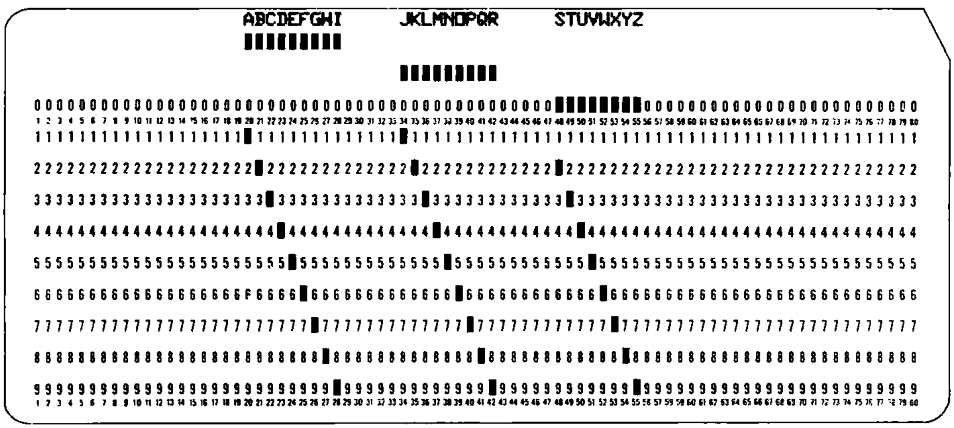
(d) There are other kinds of patterns apparent in Pascal's triangle. If the 1 at the top of the triangle is ignored, then the triangle becomes

$$
\begin{array}{ccccccccc}
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

Now the second number in each row is always the row number. Other patterns are less obvious and involve the adding and subtracting of elements. Find and describe several other patterns. You'll find this easier to do if the next two rows (obtained in parts (b) and (c)) are included in the triangle.

Data Processing Card

Patterns extend to many areas of application, as you will see throughout this chapter. Some of these applications concern computers. Data processing cards are coded according to a pattern. A typical card is shown next. It has twelve horizontal rows in which punches (perforations) can be made in order to record information. At the bottom of the card there is a row of 9's. Just above it is a row of 8's, and so on. Near the top of the card is a row of 0's. The row above the zeros, although unlabeled, is a row of 11's; the row above that is an unlabeled row of 12's. Machines record any letter of the alphabet in a vertical column by making two different punches in that column. The letter A is represented by a 12 punch and a 1 punch, abbreviated 12-1. The letter L is 11-3; Y is 0-8.



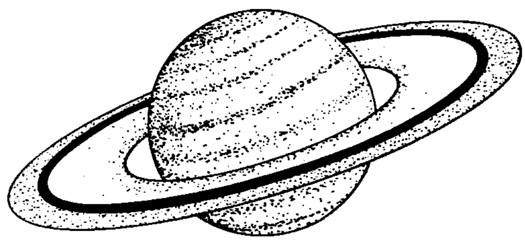
8. (a) Examine the card. Then describe the pattern for representing the letters A through I.
   (b) Describe the pattern for representing the letters J through R.
   (c) Describe the pattern for representing the letters S through Z.
   (d) Explain why "junior," which is abbreviated "JR," is the key to remembering the entire card code for letters.

SEQUENCES AND ASTRONOMY

Observing a pattern led German mathematician J. D. Titius to realize a rule for the successive distances of planets from the sun. He published the rule in 1766. Several years later, German astronomer Johann Elbert Bode called attention to Titius' rule, which has since been named *Bode's law*. First have a look at the following table. Then we'll examine the rule.

Planet	Bode's distance	Actual distance[a]
Mercury	4	3.9
Venus	7	7.2
Earth	10	10.0
Mars	16	15.2
—	28	—
Jupiter	52	52.0
Saturn	100	95.3

[a]See Application 5 for the units of measure.



Bode's law, the rule that yields the distances in the table, is based on the sequence 0, 3, 6, 12, 24, 48, 96, ... If the number 4 is added to each of the terms of this sequence, the results are Bode's distances 4, 7, 10, 16, 28, 52, 100. This sequence is Bode's law for the relative distances of planets from the sun. Perhaps you have noticed that 28 is missing in the table and you may know that there are other planets beyond Saturn, yet they are not given in the table. At the time Bode's law was first presented, only the entries shown in the table were known. What has happened since then is presented in the exercises and applications that follow.

APPLICATIONS AND EXERCISES

1. The planet Uranus was discovered by William Herschel in 1781 at a distance of 192. How does this compare with Bode's distance for the next planet beyond Saturn?