IVERSITY MATHEMATICS

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I

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UNIVERSITY MATHEMATICS I

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Preface

This is the first of two volumes that are intended to provide college and university students with a sensible continuation of the modern approach to mathematics that is being introduced in most elementary and secondary schools, with more emphasis than in the past placed on an understanding of fundamental concepts. Certain advanced topics in algebra and trigonometry, along with analytic geometry and calculus, are unified into a sequential exposition that eliminates much unnecessary duplication and is conducive to an efficient development and use of ideas and techniques. Fundamental concepts are discussed in a reasonably rigorous fashion, with adequate emphasis on important skills, and without an excess of sophistication. Many applications of mathematics have been included, and they have frequently been made the motivation for the introduction of mathematical concepts. An intuitive discussion often precedes the formal treatment of a new idea.

Although the books were written with students in engineering and the sciences in mind, they are also well suited for a good liberal arts course in mathematics. The exposition has, in the main, been kept at a level that has proved to be reasonable for the average student. However, a number of optional sections, problems, and proofs, each of which is marked by a star and may be omitted without loss of continuity, have been included as a challenge to the better students.

Important definitions, axioms, and theorems are clearly labeled, and a conscientious effort has been made to utilize each new idea and notation as frequently as possible in order to promote its intelligent use by the student. New materials and new points of view are not introduced merely for the sake of novelty, but are brought in only if they make a genuine contribution to the understanding that can be imparted to the reader.

There are several features of particular interest that we have found helpful in providing the student with a deeper understanding of elementary mathematical analysis, as well as a better background for mathematics beyond the sophomore level. First, there is the development and consistent use of the neighborhood concept in the treatment of limits. This approach gives the student a better intuitive feeling for the meaning of a limit than the more usual formal ϵ-δ attack. The second significant feature is the introduction and use of matrices for the solution of systems of linear equations and for the discussion of linear transformations in reducing a quadratic polynomial to a canonical form, as well as the application of these ideas to the solution of simple systems of differential equations. A third important feature is the use of vector algebra for the discussion of geometric ideas relating to the line and the plane in three-dimensional space, and the use of vector calculus for the development of a number of basic notions relating to curves and surfaces as well as to velocity and acceleration. The introduction and application of some elementary ideas in the calculus of complex-valued functions motivates and simplifies the use of the exponential function with an imaginary exponent.

The material in these books has been taught quite successfully for the past three years—first in the form of notes and then in an offset preliminary edition—to ordinary freshman and sophomore classes. The point of view of the exposition, the organization, and the development of the mathematical ideas, the new topics, and the intuitive development that often precedes a more rigorous formal discussion, have all been enthusiastically received by both faculty and students. We believe that this approach has enabled students to attain a desirable level of mathematical maturity in a shorter time than they could have with the more traditional approaches.

The first seven chapters of Volume I are concerned with basic ideas and the development of a consistent language and terminology for the remainder of the book. A good modern course in analytic geometry and calculus can be based on Chapters 4 and 5, the first three sections of Chapter 6, and Chapters 8 to 15 of Volume I, plus Chapters 1 to 11 of Volume II. Chapters 12 and 13 of Volume II contain adequate material for a short course in differential equations. Chapter 14 consists of an elementary treatment of the Laplace transformation, and Chapter 15 is a brief introduction to probability.

The material in Volume I can easily be covered in two five-semester-hour courses in the freshman year. The material in the first thirteen of the fifteen chapters of Volume II can be covered (with minor omissions) in two four-semester-hour courses in the sophomore year. It is, however, quite possible for a well-prepared class to complete both volumes in the two-year sequence by omitting the more elementary portions of Volume I. In order to establish the language and point of view for such students, it is advisable to study the concept of a set and the set notation in Sections 1.6, 1.7, and 1.8. The summary of Chapter 2 gives the symbols that are consistently used to denote certain special sets of numbers. Basic work on inequalities occurs in Sections 3.8 and 3.9. Chapters 4, 5, and 6, which contain the introductory work in analytic geometry and the discussion of relations and functions, should be taken in more or less detail, depending on the preparation of the class. Chapter 7, which is concerned with basic trigonometry, may be omitted for students with good high school preparation in this subject. Not more than two or three weeks is needed to cover the preceding topics, so that students with adequate high school background are then able to begin the serious work on limits and continuity in Chapter 8

We wish to thank Professors R. A. Rosenbaum and Morris Kline for their editorial suggestions, which contributed in a notable fashion to the clarity of the exposition. Many other valuable suggestions came from our colleagues in the Department of Applied Mathematics at the University of Colorado and from the long-suffering students who have seen the book through the many pains of its birth. To these students and colleagues we owe a debt that can be repaid only by the gratitude of a newer generation of students for whom the exposition has been made simpler and clearer. We are particularly appreciative of the intelligent effort put in by Mrs. Dorothy Vaughn in typing the manuscript in its many revisions. Finally, we wish to express our gratitude to our families for putting up with us during the trials and tribulations of this project.

March 1965

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Chapter 1 Fundamental Ideas

1.1 INTRODUCTION

What is mathematics? Mathematicians and philosophers have been trying for centuries—and without much success—to give a simple answer to this apparently simple question. A logician might say that mathematics is an extension of logic. A philosopher might say that mathematics is a language. A mathematician is likely to say that the question cannot be answered in any simple or concise way. Indeed, it is only by actual experience in mathematics itself that the question can be answered at all.

Most basic mathematical concepts have their roots in the physical situations that men face in their daily lives. For instance, one of the most primitive and basic of all concepts is that of counting, which is the root of the more abstract concepts of number and arithmetic. Thus, statements of the form

Two spears and three spears are five spears,

or

Two stones and three stones are five stones

have led to the more general kind of statement that

Two things and three things are five things,

or, in the most abstract and concise form,

$$2 + 3 = 5.$$

Man's ability to formulate concepts related to physical experience in short, concise "abstract" statements of this type has been the basis for his development of a civilization founded on an understanding of his environment. Much of mathematics consists of the formulation and development of abstract concepts from specific situations that arise in connection with the development of a social structure and a civilization. For example, the ancient Arab merchants developed a convenient and systematic notation as an aid to keeping track of their money. The ancient Egyptians developed many of the fundamental ideas of trigonometry so that they could relocate property lines after a flood along the Nile river bottom. In more recent times, Sir Isaac Newton was led to consider the fundamental concepts of the mathematical subject now called the calculus in order to describe the behavior of moving objects. In each case, these fundamental concepts have arisen as a result of necessity and as a supplement to our ordinary language.

Since it is clearly impossible to discuss mathematics without making use of a language, we shall consider a few fundamental concepts that are concerned with language. A language is useful only when the words in the language have a reasonably clear meaning. Thus it is important to understand just how words are given a meaning.

Without careful consideration, we might say that words are defined in terms of other words. However, it soon becomes apparent that this procedure is not always possible. For example, suppose we look in a dictionary for the meaning of the word *riddle*. The definition given may read "an enigma propounded for solution by guessing." In order to understand the meaning of *riddle* we must know the meaning of the word *enigma*. Another reference to the dictionary reveals that an enigma is "an obscure saying such as a riddle." Hence the meaning of the word *riddle* is made to depend upon itself. Just what is a riddle seems to be an enigma!

The problem of trying to define a word thus assumes a somewhat perplexing aspect. However, all is not yet lost. It is possible to give meaning to a word without using other words. This may be done by means of gesturing, pointing, and making noises. It is in just this way that young children learn the meaning of their first words. Of course, meaning does not come easily in this fashion; it is acquired only after much effort and repetition.

Nevertheless, by this crude approach a child acquires a basic vocabulary. With the aid of this vocabulary new words are defined and old words are given a refined meaning. Hence an ultimate meaning of some words is not obtained by a direct definition, but is achieved by means of a "feedback" principle. Words that are a part of the basic vocabulary of a language are often called "undefined" terms. Whatever meaning such words have is given to them by the manner in which they are allowed to be used.

Thus, the development of mathematics begins with language. New ideas are described in terms of a basic set of primitive words, the undefined terms of the language, and other words such as *which*, *the*, *for*, and so on, which are used in their customary fashion with no specific technical mathematical meaning. An example of an undefined word is the word *point*. A dictionary definition of *point* is "an undefined geometric element concerning which it is postulated that at least two exist and that two suffice to determine a straight line." The actual meaning of *point* lies in the geometry that may be built on this definition.

The feedback principle applies equally well to the learning and understanding of mathematical concepts. For example, each of us first learns the most elementary ideas associated with counting, such as addition and multiplication. Yet even after having had this knowledge for a number of years, we have never really considered the question "What is a number?" We simply use numbers, and, in doing so, we expand our knowledge of them. It is only after a consider-able length of time that one acquires the maturity necessary to understand what is *meant* by the question, a much easier task than answering it.

Accordingly, although the purpose of the early portions of this book is to give at least a partial answer to the preceding questions, we shall draw freely on illustrations from algebra and arithmetic in order to provide an intuitive approach to the abstract ideas that are introduced.

1.2 EQUALITY

Some misunderstanding has arisen from time to time in connection with the symbol for equality, $=$. It is a result of the fact that the symbol is often used in at least two (and sometimes three) different senses. In order to clarify this point we shall use the equals sign as indicated in the following definitions:

Definition 1.2a. The statement

$$a = b$$

means that a is another name for the object whose name is b.

Definition 1.2b. The statement

$$a \neq b$$

means that a and b are names of different objects.

Definition 1.2c. The statement

$$a \mathrel{.=.} b$$

means that a is *defined* to be another name for the object whose name is b.

The symbol $.=.$ may be read "is equal, by definition, to," or "is defined to be the same as."

At first sight there may appear to be no significant difference between the two notations $.=.$ and $=$, and it is true that the difference is essentially conceptual. However, this alone is sufficient reason for us to make such a distinction. An illustration based on the next definition will help to clarify this point.

Definition 1.2d. The symbol

$$[\![x]\!] \mathrel{.=.} n,$$

where n is the largest integer less than or equal to x.

The sentence of Definition 1.2d gives the bracket symbol $[\![x]\!]$ meaning. Once this meaning has been given, we may use the symbol in connection with ordinary equality. For example, we have $[\![2.34]\!] = 2$. Here the symbol $.=.$ is not used because $[\![2.34]\!]$ is not being defined as 2, but instead, $[\![2.34]\!] = 2$ by virtue of Definition 1.2d. Another example will illustrate this idea further.

Example 1.2a. Let $a_1 \mathrel{.=.} 1$, and let $a_{n+1} \mathrel{.=.} \frac{1}{2}a_n$, $n = 1, 2, 3, \ldots$. (The ellipsis customarily means "and so forth.") These statements define a sequence of numbers, denoted by $a_1, a_2, a_3, \ldots$. Which of the following statements is correct?

(a) $a_5 \doteq \frac{1}{16}$.
(b) $a_5 = \frac{1}{16}$.

The formula $a_{n+1} = \frac{1}{2}a_n$ is called a **recurrence formula** because it can be used to determine a_{n+1} when a_n is known. For example,

$$a_2 = \frac{1}{2}a_1 = \frac{1}{2} \cdot 1 = \frac{1}{2},$$

$$a_3 = \frac{1}{2}a_2 = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2^2},$$

$$a_4 = \frac{1}{2}a_3 = \frac{1}{2} \cdot \frac{1}{2^2} = \frac{1}{2^3},$$

and

$$a_5 = \frac{1}{2}a_4 = \frac{1}{2} \cdot \frac{1}{2^3} = \frac{1}{2^4} = \frac{1}{16}.$$

In each case, the expression obtained for a_n is a **consequence** of the given definition so that it is *not* correct to say that

$$a_5 \doteq \frac{1}{16},$$

but it is correct to say that

$$a_5 = \frac{1}{16}.$$

1.3 LOGIC

While it is not our purpose to make a detailed study of logic, there are certain of its principles that we need in order to develop the succeeding mathematics.

Definition 1.3a. A **proposition** is a complete declarative sentence with a definite meaning.

For example, the following sentences are all propositions:

$$2 + 3 = 5,$$
$$2 + 3 = 8,$$

Gold is a precious metal,
If it rains tonight, then tomorrow it will be fair.

A sentence of the form

x is a cat

may be interpreted in two ways. In one sense, it asserts that the 24th letter of the alphabet is a cat. In another sense, it is intended to be an incomplete sentence, where the symbol x is used to represent an arbitrary element of language. In this second sense, until x is replaced by a meaningful word or symbol, the sentence is not really complete and is therefore not to be regarded as a proposition according to Definition 1.3a.

In order to indicate that a letter represents an arbitrary element, we shall sometimes underline the symbol, as in