BESSEL FUNCTIONS FOR ENGINEERS

N. W. McLachlan

SECOND EDITION

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PREFACE TO THE SECOND EDITION

THE first edition of this book was published in 1934 as an experimental venture, when the standard of technical mathematics in colleges and universities was relatively low. There has been some improvement since 1934, but the standard is still lower than it should be. It could be raised if the subject were taught by those interested in the practical applications. In my experience, the *elements* of Bessel functions can be taught in a fourth-year undergraduate engineering course. A more extensive course, covering all the material in this book, should be given at graduate level. In the first edition, to effect simplification, the reader was expected to take a good deal for granted. After an interval of twenty years, in virtue of the improved standard of technical mathematics, as much rigour as seems to be desirable for engineers has been introduced in the present edition. The working of various problems, their meticulous correction by the teacher, and reworking by the student where necessary, is desirable to obtain results satisfactory to both parties. Each problem should be set out in the form of a technical essay in *good English*, so that the student will acquire experience in precise and logical thinking. Once the habit has been formed, it sticks!—as I know from experience.

The text of the first edition has been rewritten and extended in scope. Some of the old sections, and about half of the too numerous problems at the ends of the chapters, have been removed to make room for new subject-matter. The text and also the list of formulae has been increased by 50 per cent., the references by 30 per cent., while eight new Tables of the functions have been added. In brief, the book is essentially new. The practical applications now include the following: buckling of columns and variable struts; eddy current furnaces; loudspeaker horns; oscillation of cylinder in viscous fluid; oscillation of water in circular lake; radial temperature distribution in engine cylinder wall; resistance of solid and tubular conductors to alternating current; scattering of sound by circular cylinder; sound distribution from rigid and flexible circular disks; tapered loaded electrical transmission lines; vibration of circular membranes and plates, and of non-uniform bars; water waves in canals of variable breadth, and also of variable depth. In virtue of this comprehensive selection, it is hoped that the book will be of interest to a larger circle of readers than before.

Acknowledgements. I am indebted to the Syndics of the Cambridge University Press for permission to use entries from G. N. Watson's *Theory of Bessel Functions* to compile Tables 7, 10; to Dr. J. C. P. Miller for entries in Tables 13–16; to the Editors of the *Philosophical Magazine* for permission to reproduce Tables 29 and 30, which appeared in reference [55] in 1934; and finally to Mr. A. L. Meyers for providing the experimental data in § 8.52.

N. W. M.

June 1954

PREFACE TO THE FIRST EDITION

THE use of Bessel functions in research and design work is now so extensive that the theory and its applications may well find a place in the technical training of those destined to be engineers. The three well-known treatises by Gray, Mathews, and MacRobert, Nielsen, and Watson, whilst of great value to pure and applied mathematicians, are not suitable for engineers, whose main purpose is the application of the functions to practical problems. Consequently this volume has been written specially for engineers, so that they can become familiar with that part of the theory which is required in applied analysis. The book will be useful in connexion with Chapters II to VI of the author's treatise on *Loud Speakers*. At the same time it may serve also to introduce the functions to students reading in applied mathematics. The treatment is simple yet rigorous enough for engineers, whilst the text contains many worked examples illustrating various analytical processes. No prior knowledge, beyond that which should be obtainable in an engineering degree course, is required, and the sequence of the chapters has been arranged with this in view. The functions have not been introduced via Laplace's equation, since this approach is more in keeping with the outlook of the mathematician than with that of the engineer. For the benefit of teachers it may be said that the subject-matter has been used for a course of lectures to practising engineers with success. Owing to space limitations the theory and the more detailed parts of the practical applications have been curtailed in places, but nothing of fundamental importance to engineers has been omitted. For the same reason there is no mention of contour integrals or of Heaviside's operators. The reader who desires to supplement his knowledge can do so by aid of the reference list at the end of the book. Where necessary, practical analysis has been shorn of its technicalities, whilst references to the original works are given to enable the reader to complete his studies. Including subdivisions, there are approximately 600 examples to be worked out by the reader. Many of the examples are either practical problems or represent practical analysis dissected to effect simplification. Those devoid of a practical basis are included to facilitate understanding and memorization of the more important formulae. This is a *sine qua non* in mathematical work, since it is rather hazardous to solve practical problems with a book in one hand and a pen in the other without a proper knowledge of the processes involved. A list of important formulae is given for reference purposes, but additional formulae will be encountered in the examples. No attempt has been made to print a comprehensive set of tables, since these are available elsewhere. The tables have been selected chiefly to suit the requirements of acoustical and electrical engineers, the entries being given to four significant figures, which is ample for most engineering purposes.

The author is indebted to the Syndics of the Cambridge University Press for permission to use entries from G. N. Watson's *Bessel Functions* to complete Tables 6, 8, 9, 11, 12; to the Committee of the British Association for permission to publish Tables 17–24; also to Professor H. B. Dwight and the American Institute of Electrical Engineers for permission to reproduce Tables 25, 26. He is also indebted to Mr. R. R. M. Mallock for Tables 27, 28, giving the polar values of the ber and bei functions, which are abridgements of five-figure tables computed by him in 1928. The polar form has been suggested by Kennelly, Laws, and Pierce, who computed tables which are reproduced in the Report of the British Association, 1923. Tables are also given in *Funktionentafeln* by Jahnke and Emde. The polar form has been generalized in Chapter VIII and new formulae are developed which simplify analysis involving ber, bei, ker, and kei functions. The polar values of the ker and kei functions, viz. $N_0(z)$, $\phi_0(z)$, $N_1(z)$, $\phi_1(z)$, were not available when the manuscript was completed. They will be found in a paper by Mr. A. L. Meyers and the author, entitled 'The polar form of the ker and kei functions and its application to eddy current heating', which is published in the *Philosophical Magazine*, **18**, 610, 1934.†

Professor T. M. MacRobert has criticized the manuscript and read the proofs from the view-point of the pure mathematician, whilst Messrs. C. R. Cosens, R. R. M. Mallock, and A. L. Meyers have done likewise from the view-point of the engineer-mathematician. Mr. Meyers generously undertook the herculean task of checking the examples. The author has great pleasure in expressing his appreciation of the excellent suggestions made by these gentlemen.

† Tables 29, 30 in the present edition.

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SYMBOLS

The symbols used for various physical quantities are defined in the text. The order of a Bessel function is represented by m or n when integral, but by μ or ν when unrestricted, unless stated otherwise. The symbols for the various Bessel functions are given below:

$J_n(z)$, $J_\nu(z)$	Bessel function of the first kind.
$Y_n(z)$, $Y_\nu(z)$	,, ,, ,, second kind, as defined by Weber.
$H_n^{(1)}(z)$, $H_\nu^{(1)}(z)$	,, ,, ,, third kind.
$H_n^{(2)}(z)$, $H_\nu^{(2)}(z)$	,, ,, ,, ,, ,,
$\mathfrak{C}_n(z)$, $\mathfrak{C}_\nu(z)$	Cylinder function.
$I_n(z)$, $I_\nu(z)$	Modified Bessel function of the first kind.
$K_n(z)$, $K_\nu(z)$	,, ,, ,, ,, second kind.
$\mathbf{H}_n(z)$, $\mathbf{H}_\nu(z)$	Struve function.
$\mathbf{L}_n(z)$, $\mathbf{L}_\nu(z)$	Modified Struve function.

$S_\nu(z) = \text{ster}_\nu^2(z)+\text{stei}_\nu^2(z)$, $\psi_\nu(z) = \tan^{-1}\dfrac{\text{stei}_\nu z}{\text{ster}_\nu z}$.

$M_n(z)$, $M_\nu(z)$	$\sqrt{(\text{ber}_n^2 z+\text{bei}_n^2 z)}$, $\sqrt{(\text{ber}_\nu^2 z+\text{bei}_\nu^2 z)}$.
$\theta_n(z)$, $\theta_\nu(z)$	$\tan^{-1}\dfrac{\text{bei}_n z}{\text{ber}_n z}$, $\tan^{-1}\dfrac{\text{bei}_\nu z}{\text{ber}_\nu z}$.
$N_n(z)$, $N_\nu(z)$	$\sqrt{(\text{ker}_n^2 z+\text{kei}_n^2 z)}$, $\sqrt{(\text{ker}_\nu^2 z+\text{kei}_\nu^2 z)}$.
$\phi_n(z)$, $\phi_\nu(z)$	$\tan^{-1}\dfrac{\text{kei}_n z}{\text{ker}_n z}$, $\tan^{-1}\dfrac{\text{kei}_\nu z}{\text{ker}_\nu z}$.

$\mathbf{W}(z) = \dfrac{2}{z}\dfrac{M_1(z)}{M_0(z)}\cos(\theta_1-\theta_0-\tfrac{1}{4}\pi)$, the dissipation or loss function (Chap. VIII).

$\mathbf{\Pi}(z) = \dfrac{2}{z}\dfrac{M_1(z)}{M_0(z)}\sin(\theta_1-\theta_0-\tfrac{1}{4}\pi)$, the penetration function (Chap. VIII).

$j_{\nu,m}$ signifies the mth zero of $J_\nu(z)$.

$y_{\nu,m}$,, ,, ,, ,, $Y_\nu(z)$.

$\to \pm 0$ signifies that zero is approached from the positive or the negative side.

O means 'of order'.

$\simeq$ signifies 'approximately equal to'.

$\sim$ signifies that the right-hand side is an asymptotic formula, valid when the argument, or the order of the function, is large enough, as the case may be.

$\neq$ signifies 'is not equal to'.

$\equiv$ signifies 'is identical to'.

$R(\nu)$ signifies the real part of ν.

In dealing with the ber and bei functions (Chap. VII), the symbols $i^{\pm\frac{1}{2}}$, $i^{\pm\frac{3}{2}}$ are used to represent $e^{\pm\frac{1}{4}\pi i}$, $e^{\pm\frac{3}{4}\pi i}$, thereby avoiding the symbols $\sqrt{(i)}$, $\sqrt{(-i)}$ as these are apt to lead to confusion owing to the ambiguity in sign. Thus we take as standard functions $J_0(zi^{\frac{3}{2}}) = \text{ber}\, z+i\,\text{bei}\, z$ and $J_1(zi^{\frac{3}{2}}) = \text{ber}_1 z+i\,\text{bei}_1 z$, whereas some European writers use $J_0(z\sqrt{i}) = J_0(zi^{\frac{1}{2}}) = \text{ber}\, z-i\,\text{bei}\, z$ and $J_1(z\sqrt{i}) = J_1(zi^{\frac{1}{2}}) = -\text{ber}_1 z+i\,\text{bei}_1 z$. When using tables, care must be taken to ascertain which functions have been tabulated. The values of the various complex quantities involved are shown in Fig. 25. In analysis it is sometimes expedient to use $e^{\pm\frac{1}{2}\nu\pi i}$ instead of $i^{\pm\nu}$ to avoid possible error in the ultimate phase angle.

In works on pure mathematics some writers use amp z, whilst others prefer arg z, to signify the angle of inclination of the vector representing a complex quantity to the positive real axis. From the view-point of an engineer or a physicist, neither of these symbols says what it means, nor does it mean what it says! Amp signifies ampere or amplitude, and, if the latter, the maximum value of an oscillation is implied, which corresponds to the modulus of the complex quantity, but not to an angle. Arg z implies the independent variable or argument of a function, such as z in $J_n(z)$. Consequently, to avoid ambiguity, we shall use 'phase z' or θ to denote the angle of the vector with the positive real axis.

$\int^{z}$ signifies an indefinite integral.

CONTENTS

I

SHORT HISTORICAL INTRODUCTION; FUNCTIONS OF ORDER ZERO; APPLICATIONS

1.10. Introduction

Bessel functions, like many other branches of mathematics, had their origin in the solution of physical problems. Particular cases of these functions occurred in the solutions of differential equations over a century before the advent of the famous memoir written by the German astronomer F. W. Bessel in 1824 and published in 1826. Consequently, the nomenclature 'Bessel Function' did not exist prior to 1826.

In 1732 Daniel Bernoulli, a Swiss mathematician well known for his theorem in hydraulics, studied the problem of the oscillation of a heavy flexible chain suspended with its lower end free. He obtained a differential equation of the type

$$\frac{d^2y}{dz^2}+\frac{1}{z}\frac{dy}{dz}+\frac{k^2y}{z}=0, \tag{1}$$

which may be transformed to the same form as that given by Bessel about a century later. Bernoulli's problem is considered in §§ 5.20–5.24. L. Euler, also a Swiss mathematician of the eighteenth century, so well known to engineers for his long strut formula, investigated the vibration of a stretched circular membrane in 1764, and obtained a differential equation identical in form with that now accepted as the generalized Bessel equation. He studied Bernoulli's problem also in 1781, and calculated some of the early zeros pertaining to the first solution of (1). During the solution of an astronomical problem in 1770, the French mathematician J. L. de Lagrange arrived at an equation whose solution, presented in the form of an infinite series, involved coefficients now coupled with Bessel's name. An investigation of the coefficients obtained by Lagrange was conducted some years later by the Italian mathematician F. Carlini and the French analyst P. S. de Laplace.

The year 1822 is of singular importance in the history of mathematics owing to the publication of J. B. J. Fourier's treatise on *The Analytical Theory of Heat*. This epoch-making work, which was 'crowned' in 1812, had adorned the archives of the Paris Academy of

Sciences for about eleven years. Its publication was delayed for fear that it should affect adversely the prestige of the powers that were. In treating a problem on the distribution of temperature in a cylinder, which was heated and then allowed to cool under certain conditions, Fourier obtained a particular case of a Bessel equation and gave its solution (zero order). Further analytical researches on the distribution of temperature in spheres and cylinders were published by the French mathematician, S. D. Poisson, so well known to engineers for his ratio $\sigma = \dfrac{\text{lateral strain}}{\text{longitudinal strain}}$. Poisson's work was associated with functions of Bessel type.

To sum up the situation: prior to 1824, various particular cases of a certain differential equation were investigated by mathematicians, but no attempt was made to deal with these equations in a systematic way, and the terminology 'Bessel Functions' did not exist.

1.20. Bessel's coefficients

In 1824 F. W. Bessel studied a problem associated with elliptic planetary motion. He found that an astronomical quantity, termed the 'eccentric anomaly' θ, could be represented by an infinite series of the form

$$\theta = \chi + A_1 \sin\chi + A_2 \sin 2\chi + \dots \tag{1}$$

$$= \chi + \sum_{n=1}^{\infty} A_n \sin n\chi, \tag{2}$$

where

$$\chi = (\theta - x \sin\theta). \tag{3}$$

The coefficients $A_1,\dots, A_n$ may be obtained by a process resembling that used in the Fourier analysis of alternating currents. Assuming that the right-hand side of (1) may be differentiated term by term, we get

$$d\theta = d\chi(1 + A_1 \cos\chi + 2A_2 \cos 2\chi + \dots). \tag{4}$$

Multiplying each side by $\cos n\chi$, and integrating over the range $(0, 2\pi)$, yields

$$\int_0^{2\pi} \cos n\chi \, d\theta = n\pi A_n, \tag{5}$$

since all integrals on the right-hand side vanish, except that involving $\cos^2 n\chi$. Thus

$$A_n = (1/n\pi) \int_0^{2\pi} \cos n\chi \, d\theta. \tag{6}$$

Now $\chi = (\theta - x\sin\theta)$, so (6) may be written

$$A_n = (1/n\pi)\int_0^{2\pi} \cos(n\theta - z\sin\theta)\,d\theta, \tag{7}$$

where $z = nx$, $n = 1, 2, 3, \ldots$.

If we put $A_n = (2/n)J_n(z)$, we obtain Bessel's definition of the functions which bear his name, i.e.

$$J_n(z) = (1/2\pi)\int_0^{2\pi} \cos(n\theta - z\sin\theta)\,d\theta. \tag{8}$$

This function satisfies the differential equation

$$\frac{d^2y}{dz^2} + \frac{1}{z}\frac{dy}{dz} + \left(1 - \frac{n^2}{z^2}\right)y = 0, \tag{9}$$

or

$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} + (z^2 - n^2)y = 0, \tag{10}$$

which is obtained on multiplying (9) throughout by z^2.

We shall now show in the simple case $n = 0$, that (8) satisfies (9), and, therefore, (10) also.

Take

$$2\pi J_0(z) = y = \int_0^{2\pi} \cos(z\sin\theta)\,d\theta; \tag{11}$$

then

$$dy/dz = -\int_0^{2\pi} \sin(z\sin\theta)\sin\theta\,d\theta = \int_0^{2\pi} \sin(z\sin\theta)\,d(\cos\theta) \tag{12}$$

$$= \big[\cos\theta\sin(z\sin\theta)\big]_0^{2\pi} - z\int_0^{2\pi} \cos(z\sin\theta)\cos^2\theta\,d\theta, \tag{13}$$

so

$$y'/z = -\int_0^{2\pi} \cos(z\sin\theta)\cos^2\theta\,d\theta. \tag{14}$$

Also, by (12),

$$y'' = -\int_0^{2\pi} \cos(z\sin\theta)\sin^2\theta\,d\theta. \tag{15}$$

Hence from (11), (14), (15)

$$(1/2\pi)\{y'' + y'/z + y\} = (1/2\pi)\int_0^{2\pi} \cos(z\sin\theta)\{-(\sin^2\theta + \cos^2\theta) + 1\}\,d\theta = 0 \tag{16}$$

as required.

(9) is known as Bessel's equation for functions of integral order n. It is a linear differential equation of the second order having variable

coefficients, namely $1/z$ and $(1-n^2/z^2)$. By the theory of linear differential equations, it has two distinct or linearly independent solutions, i.e. one is not a constant multiple of the other. $J_n(z)$ is taken as the first solution, and the second will be introduced later on. In virtue of its relation to the A_n in (1), $J_n(z)$ is sometimes called a Bessel coefficient, but it is regarded more generally as a Bessel function of the first kind of order n.

1.21. First solution of Bessel's equation

To illustrate the method of solution, we choose the simple case where n in (9), § 1.20, is zero. The equation to be solved is then

$$\frac{d^2y}{dz^2}+\frac{1}{z}\frac{dy}{dz}+y = 0. \tag{1}$$

Following Frobenius, we assume that y may be represented by an infinite power series of the form (see Appendix III)

$$y = z^\mu\{a_0+a_1 z+a_2 z^2+a_3 z^3+...\}, \tag{2}$$

where μ and the coefficients $a_0, a_1, a_2,...$ are to be determined. This series is to be substituted into (1), and the series obtained by adding the three sets of terms is to be equated to zero. Assuming for the present that term by term differentiation of (2) is valid, we obtain

$$y = z^\mu\{a_0+a_1 z+a_2 z^2+...\},$$

$$\frac{1}{z}\frac{dy}{dz} = z^\mu\{a_0 \mu z^{-2}+a_1(\mu+1)z^{-1}+a_2(\mu+2)+ +a_3(\mu+3)z+a_4(\mu+4)z^2+...\},$$

$$\frac{d^2y}{dz^2} = z^\mu\{a_0(\mu-1)\mu z^{-2}+a_1 \mu(\mu+1)z^{-1}+a_2(\mu+1)(\mu+2)+ +a_3(\mu+2)(\mu+3)z+a_4(\mu+3)(\mu+4)z^2+...\}.$$

If the sum of the three series is to vanish, the sum of the coefficients of like powers of z must vanish also. Equating coefficients to zero, we have $a_0 \mu^2 = 0$, $a_1(\mu+1)^2 = 0$, $a_0+a_2(\mu+2)^2 = 0$, $a_1+a_3(\mu+3)^2 = 0$, and so on. In the first case $a_0 \mu^2 = 0$,† so either $a_0 = 0$ or $\mu = 0$. For the present, however, we shall not take either a_0 or μ to be zero, as the results obtained in this way will be useful later on. In the second case $a_1(\mu+1)^2 = 0$, so $a_1 = 0$ provided $\mu \neq -1$. Similarly

$$a_3 = a_5 = a_7 = ... = 0,$$

† This is known as the indicial equation, since it serves to determine the index μ.

while
$$a_2 = -a_0/(\mu+2)^2, \qquad a_4 = a_0/(\mu+2)^2(\mu+4)^2,$$
$$a_6 = -a_0/(\mu+2)^2(\mu+4)^2(\mu+6)^2,$$
and so on. Inserting the various coefficients into (2) and writing y_1 for y, we get
$$y_1 = a_0 z^{\mu}\left\{1-\frac{z^2}{(\mu+2)^2}+\frac{z^4}{(\mu+2)^2(\mu+4)^2}-\frac{z^6}{(\mu+2)^2(\mu+4)^2(\mu+6)^2}+...\right\}. \tag{3}$$
We have seen that if y_1 is to satisfy (1), either a_0 or μ must be zero. Hence, if we put $\mu = 0$ and $a_0 = 1$ in (3), we obtain
$$y_1 = J_0(z) = \left\{1-(\tfrac{1}{2}z)^2+\frac{(\frac{1}{2}z)^4}{(2!)^2}-\frac{(\frac{1}{2}z)^6}{(3!)^2}+...\right\} \tag{4}$$
$$= \left\{1-\frac{z^2}{2^2}+\frac{z^4}{2^2.4^2}-\frac{z^6}{2^2.4^2.6^2}+...\right\} = \sum_{r=0}^{\infty}(-1)^r\frac{(\frac{1}{2}z)^{2r}}{(r!)^2}. \tag{5}$$

This series and its derivatives are absolutely convergent for all finite values of z real or complex, and *uniformly* convergent in any bounded region of the z-plane (see Appendix II). It represents $J_0(z)$, the first solution of (1), and is defined to be Bessel's function of the *first* kind of order zero. Since $J_0(z) = J_0(-z)$, it is an even function of z. In virtue of *uniform* convergence, $J_0(z)$ and its derivatives are continuous functions of z finite, term by term differentiation of their series representations being valid. If in (8), § 1.20, we put $n = 0$, then expand the integrand and integrate term by term (see Appendix II, § 4), (4) above is reproduced.

1.22. Zeros

An alternating *function* of z real has zeros at intervals which need not be equal. It may be represented sometimes by a power series having terms with both positive and negative signs.† For instance, if z is real, $\cos z$, which is represented by the alternating series
$$1-\frac{z^2}{2!}+\frac{z^4}{4!}-..., \tag{1}$$
is an alternating *periodic* function‡ having amplitude unity. An alternating series, however, does not necessarily represent an alternating

† If all terms have the same sign in a given range of z, the function cannot alternate therein.

‡ Repeats itself *exactly* at a constant interval.

function. The exponential function $e^{-\frac{1}{2}z}$, which is represented by the alternating series

$$1-\tfrac{1}{2}z+\frac{(\frac{1}{2}z)^2}{2!}-\frac{(\frac{1}{2}z)^3}{3!}+\dots, \tag{2}$$

is monotonic (devoid of oscillation), and tends to zero as $z \to +\infty$. Squaring each term and keeping the original signs, we get

$$1-\frac{z^2}{2^2}+\frac{z^4}{2^2.4^2}-\frac{z^6}{2^2.4^2.6^2}+\dots = J_0(z). \tag{3}$$

Now the series in (1), (3) both have the form $\sum_{r=0}^{\infty}(-1)^r a_r z^{2r}$, and since $\cos z$ vanishes when z has certain (real) values, we ask whether $J_0(z)$ has this property too. We accept the result proved in [124], that $J_0(z)$ has an infinity of *simple*† real zeros.

It is interesting to find that by squaring each term of the alternating series (2), which represents a positive monotonic function, and preserving the original signs, a series is obtained which represents a function having an infinite number of zeros.

1.23. Computation of the first zeros of $J_0(z)$

Neglecting all but the first three terms in (3), § 1.22, we put tentatively

$$z^4/(2^2.4^2)-z^2/2^2+1 = 0, \tag{1}$$

so that

$$z^4-16z^2+64 = (z^2-8)^2 = 0. \tag{2}$$

Thus

$$z \simeq \pm 2{\cdot}828, \tag{3}$$

and these values are a first approximation to the first zeros of $J_0(z)$.‡ To obtain a second approximation, we now include the term in z^6 from (3), § 1.22. Writing $y = z^2/4$, we get the equation

$$y^3-9y^2+36y-36 = 0. \tag{4}$$

Inserting $y = 1{\cdot}4$, $1{\cdot}5$ in succession, the left-hand side has the respective values $-0{\cdot}504$, $1{\cdot}125$, so one root of (4) lies between $y = 1{\cdot}4$ and $1{\cdot}5$. Referring to Fig. 1, by interpolation

$$x = 0{\cdot}1\times 0{\cdot}504/(1{\cdot}125+0{\cdot}504) \simeq 0{\cdot}0309. \tag{5}$$

Then the value of the root is

$$y \simeq 1{\cdot}4309, \tag{6}$$

so

$$z \simeq \pm 2(1{\cdot}4309)^{\frac{1}{2}} \simeq \pm 2{\cdot}392, \tag{7}$$

which is a second approximation to the first zeros of $J_0(z)$.

To obtain a third approximation, we employ Taylor's theorem for $|h|$ small in the abbreviated form

$$J_0(z+h) \simeq J_0(z)+hJ_0'(z). \tag{8}$$

† The graph crosses the z-axis, but is never tangential to it.

‡ Applying this procedure to (1), § 1.22, yields $z \simeq \pm 1{\cdot}59$, whereas the true values are $\pm\frac{1}{2}\pi \simeq \pm 1{\cdot}5708$. The accuracy exceeds that in (3), because $z \simeq \pm 1{\cdot}59$ compared with $\pm 2{\cdot}828$. In the latter case the terms omitted from the series (3), § 1.22, are, therefore, of greater relative importance than those omitted from (1), § 1.22.