Alexander Melnikov

Risk Analysis in Finance and Insurance

Second Edition

Chapman & Hall/CRC FINANCIAL MATHEMATICS SERIES

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CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business
A CHAPMAN & HALL BOOK

Chapman & Hall/CRC
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

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Printed in the United States of America on acid-free paper
10 9 8 7 6 5 4 3 2 1

International Standard Book Number: 978-1-4200-7052-1 (Hardback)

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Preface to 2nd edition

This book deals with the notion of "risk" and is devoted to analysis of risks in finance and insurance. We will study risks associated with financial and insurance contracts, by which we understand *risks* to be uncertainties that may result in financial loss and affect the ability to make payments associated with the corresponding contracts. Our approach to this analysis is based on the development of a methodology for estimating the present value of the future payments given current financial, insurance, and other information. Using this approach, one can appropriately define notions of *price* for a financial contract, of *premium* for insurance policy, and of *reserve* for an insurance company. Historically, financial risks were subject to elementary mathematics of finance and were treated separately from insurance risks, which were analyzed in actuarial science. The development of quantitative methods based on stochastic analysis is a key achievement of modern financial mathematics. These methods can be naturally extended and applied in the area of actuarial science, thus leading to unified methods of risk analysis and management.

The aim of this book is to give an accessible comprehensive introduction to the main ideas, methods, and techniques that transform risk management into a quantitative science. Because of the interdisciplinary nature of this book, many important notions and facts from mathematics, finance, and actuarial science are discussed in an appropriately simplified manner. Our goal is to present interconnections among these disciplines and to encourage our reader toward further study of the subject. We indicate some initial directions in the Bibliographic Remarks.

This edition is reorganized in a way that allows a natural flow of topics covered in the first edition to be combined together with new additions such as: financial markets with stochastic volatility, risk measures, risk-adjusted performance measures, equity-linked insurance, and so forth. The substantial extension of the section regarding the foundations of Probability and Stochastic Analysis makes this book self-contained. Furthermore, an increased number of worked examples and a collection of some 140 problems, which is accompanied by the Instructor's Solutions Manual, make this edition more attractive both from a research and a pedagogical perspective. This book can be readily used as a textbook for a Mathematical Finance course, both at introductory undergraduate and advanced graduate levels. It has been used for teaching Mathematical Finance at both levels at the University of Alberta, and many student comments and recommendations are taken into account in this edition.

The author thanks his graduate students Anna Evstafyeva, Hao Li, and Henry Heung for their help in introducing some new worked examples and problems in this edition. The author is also very grateful to Dr. Alexei Filinkov for translating, editing, and preparing the manuscript.

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October 2010

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Introduction

Financial and insurance markets always operate under various types of uncertainties that can affect the financial position of companies and individuals. In financial and insurance theories, these uncertainties are usually referred to as risks. Given certain states of the market, and the economy in general, one can talk about risk exposure. It is expected that individuals, companies, and public establishments that aim to accumulate wealth should examine their risk exposure. The process of risk management consists of a sequence of corresponding actions over a period of time that are designed to mitigate the level of risk exposure. Some of the main principles and ingredients of risk management are qualitative identification of risk, estimation of possible losses, choosing the appropriate strategies for avoiding losses and for shifting the risk to other parts of the financial system, including analysis of the involved costs, and using feedback for developing adequate controls.

The first six chapters of this book are devoted to the financial market risks. We aim to give an elementary and yet comprehensive introduction to the main ideas, methods, and stochastic models of financial mathematics. The probabilistic approach appears to be one of the most efficient ways of modeling uncertainties in financial markets. Risks (or uncertainties of financial market operations) are described in terms of statistically stable stochastic experiments, and therefore estimation of risks is reduced to the construction of financial forecasts adapted to these experiments. Using conditional expectations, one can quantitatively describe these forecasts given the observable market prices and events. Thus, it can be possible to construct dynamic hedging strategies and those for optimal investment. The foundations and key concepts of the modern methodology of quantitative financial analysis are the main focus of Chapters 1–6.

Insurance against possible financial losses is one of the key ingredients of risk management. However, the insurance business is an integral part of the financial system. Chapters 7–8 focus on the problems of managing insurance risks. Multiple intrinsic connections between insurance risks and financial risks are also considered.

Our treatment of insurance risk management demonstrates that methods of risk evaluation and management in insurance and finance are interrelated and can be treated using a single integrated approach. Estimations of future payments and their corresponding risks are key operational tasks of financial and insurance companies. Management of these risks requires an accurate evaluation of the present values of future payments, and therefore the adequate modeling of (financial and insurance) risk processes. Stochastic analysis is one of the most powerful tools for such modeling, and it is the fundamental basis of our presentation.

Finally, we note that probabilistic methods were used in finance and insurance since the early 1950s. They were developed extensively over the past decades, especially after the seminal papers by F. Black and M. Scholes and R. C. Merton, published in 1973.

Contents

Chapter 1

Introductory Concepts of Financial Risk Management and Related Mathematical Tools

1.1 Introductory concepts of the securities market

The notion of an *asset* (anything of value) is one of the fundamental notions in mathematical finance. Assets can be *risky* and *non-risky*. Here *risk* is understood to be an uncertainty that can cause loss (e.g., of wealth). The most typical representatives of such assets are the *basic securities*: *stocks* S and *bonds* (bank accounts) B. These securities constitute the basis of a *financial market* that can be understood as a space of assets equipped with a structure for their trading.

Stocks are share securities that accumulate capital required for a company's successful operation. The stockholder holds the right to participate in the control of the company and to receive dividends.

Bonds are debt securities issued by a government or a company for accumulating capital, restructuring debts, and so forth. In contrast to stocks, bonds are issued for a specified period of time. The essential characteristics of a bond include their exercise and maturity time, *face value* (principal or nominal), *coupons* (payments up to maturity), and *yield* (return up to maturity). The zero-coupon bond is similar to a bank account, and its yield corresponds to a bank interest rate.

An *interest rate* $r \geq 0$ is typically quoted by banks as an annual percentage. Suppose that a client opens an account with a deposit of B_0, then at the end of a 1-year period the client's non-risky profit is $\Delta B_1 = B_1 - B_0 = rB_0$. After n years, the balance of this account will be $B_n = B_{n-1} + rB_0$, given that only the initial deposit B_0 is reinvested every year. In this case, r is referred to as a *simple interest*.

Alternatively, the earned interest also can be reinvested (compounded), and then, at the end of n years, the balance will be $B_n = B_{n-1}(1+r) = B_0(1+r)^n$. Note that here the ratio $\Delta B_n / B_{n-1}$ reflects the profitability or return of the investment as it is equal to r, the *compound interest*.

Now suppose that interest is compounded m times per year, then

$$B_n = B_{n-1}\left(1+\frac{r^{(m)}}{m}\right)^m = B_0\left(1+\frac{r^{(m)}}{m}\right)^{mn}.$$

Such rate $r^{(m)}$ is quoted as a *nominal* (annual) interest rate, and the equivalent *effective* (annual) interest rate is equal to $r = \left(1+\frac{r^{(m)}}{m}\right)^m - 1$.

Let $t \geq 0$, and consider the ratio

$$\frac{B_{t+\frac{1}{m}} - B_t}{B_t} = \frac{r^{(m)}}{m},$$

where $r^{(m)}$ is a nominal annual interest rate compounded m times per year. Then another rate

$$r = \lim_{m\to\infty} \frac{B_{t+\frac{1}{m}} - B_t}{\frac{1}{m}B_t} = \lim_{m\to\infty} r^{(m)} = \frac{1}{B_t}\frac{dB_t}{dt}$$

is called the nominal annual interest rate *compounded continuously*. Clearly, $B_t = B_0 e^{rt}$.

Thus, the concept of interest is one of the essential components in the description of money value time evolution. Now consider a series of periodic payments (deposits) $f_0, f_1, \ldots, f_n$ (*annuity*). It follows from the formula for compound interest that the present value of k-th payment is equal to $f_k\left(1+r\right)^{-k}$, and therefore the present value of the annuity is $\sum_{k=0}^{n} f_k\left(1+r\right)^{-k}$.

Worked Example 1.1 *Let an initial deposit into a bank account be* \$10,000. *Given that* $r^{(m)} = 0.1$, *find the account balance at the end of 2 years for* $m = 1, 3,$ *and* 6. *Also find the balance at the end of each of years* 1 *and* 2 *if the interest is compounded continuously at the rate* $r = 0.1$.

Solution Using the notion of compound interest, we have

$$B_2^{(1)} = 10,000\left(1+0.1\right)^2 = 12,100$$

for interest compounded once per year;

$$B_2^{(3)} = 10,000\left(1+\frac{0.1}{3}\right)^{2\times 3} \approx 12,174$$

for interest compounded three times per year;

$$B_2^{(6)} = 10,000\left(1+\frac{0.1}{6}\right)^{2\times 6} \approx 12,194$$

for interest compounded six times per year.

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For interest compounded continuously we obtain

$$B_1^{(\infty)} = 10,000\, e^{0.1} \approx 11,052\,, \qquad B_2^{(\infty)} = 10,000\, e^{2\times 0.1} \approx 12,214\,. \square$$

Stocks are significantly more volatile than bonds, and therefore they are characterized as *risky assets*. Similarly to bonds, one can define their *profitability* or *return* $\rho_n = \Delta S_n/S_{n-1}$, $n = 1, 2, \ldots$, where S_n is the price of a stock at time n. Then we have the following discrete equation for stock prices: $S_n = S_{n-1}(1 + \rho_n)$, $S_0 > 0$.

The mathematical model of a financial market formed by a bank account B (with an interest rate r) and a stock S (with profitabilities ρ_n) is referred to as a (B, S)-market.

The volatility of prices S_n is caused by a great variety of sources, some of which may not be easily observed. In this case, the notion of *randomness* appears to be appropriate, so that S_n, and therefore ρ_n, can be considered as *random variables.* Since at every time step n the price of a stock goes either up or down, then it is natural to assume that profitabilities ρ_n form a sequence of independent random variables $(\rho_n)_{n=1,2,\ldots}$ that take values b and a $(b > a)$ with probabilities p and q, respectively $(p + q = 1)$. Next, we can write ρ_n as a sum of its mean $\mu = bp + aq$ and a random variable $w_n = \rho_n - \mu$, which has the expectation equal to zero. Thus, profitability ρ_n can be described in terms of an independent random deviation w_n from the mean profitability μ.

When the time steps become smaller, the oscillations of profitability become more chaotic. Formally, the limit continuous model can be written as

$$\frac{\dot{S}_t}{S_t} \equiv \frac{dS_t}{dt}\frac{1}{S_t} = \mu + \sigma \dot{W}_t\,,$$

where μ is the mean profitability, σ is the volatility of the market, and $\dot{W}_t$ is the Gaussian white noise.

The formulas for compound and continuous interest rates together with the corresponding equation for stock prices define the binomial (Cox-Ross-Rubinstein) and the diffusion (Black-Scholes) models of the market, respectively.

A participant in a financial market usually invests free capital in various available assets that then form an *investment portfolio*. The effective management of the capital is realized through a process of building and managing the portfolio. The redistribution of a portfolio with the goal of limiting or minimizing the risk in various financial transaction is usually referred to as *hedging*. The corresponding portfolio is then called a *hedging portfolio*. An investment strategy (portfolio) that may give a profit even with zero initial investment is called an *arbitrage* strategy. The presence of arbitrage reflects the instability of a financial market.

The development of a financial market offers the participants *derivative securities*, that is, securities that are formed on the basis of the basic securities – stocks and bonds. The derivative securities (forwards, futures, options, etc.) require smaller initial investment and play the role of insurance against possible losses. They also increase the liquidity of the market.

For example, suppose company A plans to purchase shares of company B at the end of the year. To protect itself from a possible increase in shares prices, company A reaches an agreement with company B to buy the shares at the end of the year for a fixed (forward) price F. Such an agreement between the two companies is called a *forward contract* (or simply, *forward*).

Now suppose that company A plans to sell some shares to company B at the end of the year. To protect itself from a possible fall in price of those shares, company A buys a *put option* (seller's option), which confers the right to sell the shares at the end of the year at the fixed *strike price* K. Note that, in contrast to the forwards case, a holder of an option must pay a *premium* to its issuer.

Futures contract is an agreement similar to the forward contract, but the trading takes place on a *stock exchange*, a special organization that manages the trading of various goods, financial instruments, and services.

Finally, we reiterate here that mathematical models of financial markets, methodologies for pricing various financial instruments and for constructing optimal (minimizing risk) investment strategies, are all subject to modern mathematical finance.

1.2 Probabilistic foundations of financial modeling and pricing of contingent claims

Suppose that a non-risky asset B and a risky asset S are completely described at any time $n = 0, 1, 2, \ldots$ by their prices. Therefore, it is natural to assume that the price dynamics of these securities are the essential component of a financial market. These dynamics are represented by the following equations:

$$\begin{aligned} \Delta B_n &= r B_{n-1}\,, \quad B_0 = 1\,, \\ \Delta S_n &= \rho_n S_{n-1}\,, \quad S_0 > 0\,, \end{aligned}$$

where $\Delta B_n = B_n - B_{n-1}$, $\Delta S_n = S_n - S_{n-1}$, $n = 1, 2, \ldots$; $r \geq 0$ is a constant rate of interest and ρ_n represent the level of risk. Quantities ρ_n will be specified later in this section.

Another important component of a financial market is the set of admissible actions or strategies that are allowed in dealing with assets B and S. A sequence $\pi = (\pi_n)_{n=1,2,\ldots} \equiv (\beta_n, \gamma_n)_{n=1,2,\ldots}$ is called an *investment strategy* (portfolio) if for any $n = 1, 2, \ldots$, the quantities β_n and γ_n are determined by prices $S_1, \ldots, S_{n-1}$. In other words, $\beta_n = \beta_n(S_1, \ldots, S_{n-1})$ and $\gamma_n = \gamma_n(S_1, \ldots, S_{n-1})$ are functions of $S_1, \ldots, S_{n-1}$, and they are interpreted as the amounts of assets B and S, respectively, at time n. The *value* of a portfolio π is

$$X_n^\pi = \beta_n B_n + \gamma_n S_n ,$$

where $\beta_n B_n$ represents the part of the capital deposited in a bank account and $\gamma_n S_n$ represents the investment in shares. If the value of a portfolio can change only because of changes in assets prices $\Delta X_n^\pi = X_n^\pi - X_{n-1}^\pi = \beta_n \Delta B_n + \gamma_n \Delta S_n$, then π is said to be a *self-financing* portfolio. The class of all such portfolios is denoted SF.

A common feature of all derivative securities in a (B, S)-market is their potential liability (payoff) f_N at a future time N. For example, for forwards, we have $f_N = S_N - F$ and for call options $f_N = (S_N - K)^+ \equiv \max\{S_N - K, 0\}$. Such liabilities inherent in derivative securities are called *contingent claims.* One of the most important problems in the theory of contingent claims is their *pricing* at any time before the expiry date N. This problem is related to the problem of *hedging contingent claims.* A self-financing portfolio is called a *hedge* for a contingent claim f_N if $X_N^\pi \geq f_N$ for any behavior of the market. If a hedging portfolio is not unique, then it is important to find a hedge π^* with the minimum value $X_n^{\pi^*} \leq X_n^\pi$ for any other hedge π. Hedge π^* is called the *minimal hedge.* The minimal hedge gives an obvious solution to the problem of pricing a contingent claim: the fair price of the claim is equal to the value of the minimal hedging portfolio. Furthermore, the minimal hedge manages the risk inherent in a contingent claim.

Next, we introduce some basic notions from probability theory and stochastic analysis that are helpful in studying risky assets. We start with the fundamental notion of an "experiment" when the set of possible outcomes of the experiment is known, but it is not known *a priori* which of those outcomes will take place (this constitutes the *randomness* of the experiment).

Example 1.1 (Trading on a stock exchange)

A set of possible exchange rates between the dollar and the euro is always known before the beginning of trading but not the exact value. □

Let Ω be the set of all elementary outcomes ω and let $\mathcal{F}$ be the set of all *events* (non-elementary outcomes), which contains the *impossible* event $\emptyset$ and the *certain* event Ω.

Next, suppose that after repeating an experiment n times, an event $A \in \mathcal{F}$ occurred n_A times. Let us consider random experiments that have the following property of *statistical stability:* for any event A, there is a number $P(A) \in [0, 1]$ such that $n_A/n \to P(A)$ as $n \to \infty$. This number $P(A)$ is called the *probability* of event A. Probability $P : \mathcal{F} \to [0, 1]$ is a function with the following properties:

1. $P(\Omega) = 1$ and $P(\emptyset) = 0$;

2. $P(\cup_k A_k) = \sum_k P(A_k)$ for $A_i \cap A_j = \emptyset$.

The triple $(\Omega, \mathcal{F}, P)$ is called a *probability space.* For the rest of this section, we assume that the set Ω is countable. In this case, $(\Omega, \mathcal{F}, P)$ is referred to as a *discrete probability space.*

Every event $A \in \mathcal{F}$ can be associated with its *indicator*:

$$I_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{if } \omega \in \Omega \setminus A \end{cases}.$$

Any function $X : \Omega \to \mathbb{R}$ is called a *random variable.* An indicator is the simplest example of a random variable. Any random variable X on a discrete probability space is *discrete* since the range of function $X(\cdot)$ is countable: $(x_k)_{k=1,2,\ldots}$. In this case, we have the following representation:

$$X(\omega) = \sum_{k=1}^{\infty} x_k I_{A_k}(\omega),$$

where $A_k \in \mathcal{F}$ and $\cup_k A_k = \Omega$. A discrete random variable X is called *simple* if the corresponding sum is finite. The function

$$F_X(x) := P(\{\omega : X \le x\}), \quad x \in \mathbb{R} \tag{1.1}$$

is called the *distribution function* (or *cumulative distribution function*) of X. For a discrete X, we have

$$F_X(x) = \sum_{k: x_k \le x} P(\{\omega : X = x_k\}) \equiv \sum_{k: x_k \le x} p_k. \tag{1.2}$$

The sequence $(p_k)_{k=1,2,\ldots}$ is called the *probability distribution* of a discrete random variable X, and we have $\sum_k p_k = 1$.

Note the following properties of the distribution function:

(D1) $F_X(x)$ are non-decreasing functions of x;

(D2) $F_X(x) \searrow 0$ as $x \to -\infty$ and $F_X(x) \nearrow 1$ as $x \to +\infty$.

The *expectation* (*expected value* or *mean value*) of X is

$$E(X) = \sum_{k \ge 1} x_k p_k.$$

Given a random variable X, for most functions $g : \mathbb{R} \to \mathbb{R}$ it is possible to define a random variable $Y = g(X)$ with expectation

$$E(Y) = \sum_{k \ge 1} g(x_k) p_k.$$

In particular, for any $k = 1, 2, \ldots$, the quantity $E(X^k)$ is called the k-th *moment* of X, and the quantity

$$Var(X) = E\Big[\big(X - E(X)\big)^2\Big]$$

is called the *variance* of X.

Note the following straightforward properties:

1. *Linearity of the expectation:* for any random variables $X_1, \ldots, X_n$ and any constants $c_1, \ldots, c_n$, we have

$$E\Big(\sum_{i=1}^{n} c_i\, X_i\Big) = \sum_{i=1}^{n} c_i\, E(X_i)\,;$$

2. For any random variable X and any constant c,

$$Var(c\, X) = c^2\, Var(X)\,.$$

Example 1.2 (Examples of discrete probability distributions)

1. Bernoulli:

$$p_0 = P(\{\omega :\ X = a\}) = p\,, \quad p_1 = P(\{\omega :\ X = b\}) = 1 - p\,,$$

where $p \in [0,1]$ and $a, b \in \mathbb{R}$.

2. Binomial:

$$p_m = P(\{\omega :\ X = m\}) = \binom{n}{k} p^m (1-p)^{n-m}\,,$$

where $p \in [0,1]$, $n \geq 1$ and $m = 0, 1, \ldots, n$.

3. Poisson (with parameter $\lambda > 0$):

$$p_m = P(\{\omega :\ X = m\}) = e^{-\lambda} \frac{\lambda^m}{m!}$$

for $m = 0, 1, \ldots$.

Consider a positive random variable $\widetilde{Z}$ on a probability space $(\Omega, \mathcal{F}, P)$. Suppose that $E(\widetilde{Z}) = 1$, then, for any event $A \in \mathcal{F}$, define its new probability:

$$\widetilde{P}(A) = E(\widetilde{Z} I_A)\,. \tag{1.3}$$

The expectation of a random variable X with respect to this new probability is $\widetilde{E}(X) = E(\widetilde{Z} X)$, and this rule is referred to as *change of the probability measure under the expectation sign.* Random variable $\widetilde{Z}$ is called the *density* of the probability $\widetilde{P}$ with respect to P. The proof of this formula is based on the linearity of the expectation:

$$\begin{aligned}
\widetilde{E}(X) &= \sum_k x_k \widetilde{P}(\{\omega :\ X = x_k\}) = \sum_k x_k E\big(\widetilde{Z}\, I_{\{\omega:\ X = x_k\}}\big) \\
&= \sum_k E\big(\widetilde{Z}\, x_k\, I_{\{\omega:\ X = x_k\}}\big) = E\Big(\widetilde{Z} \sum_k x_k\, I_{\{\omega:\ X = x_k\}}\Big) \\
&= E(\widetilde{Z} X)\,.
\end{aligned}$$

For discrete random variables X and Y with values $(x_i)_{i=1,2,\dots}$ and $(y_i)_{i=1,2,\dots}$, respectively, consider the probabilities

$$P(\{\omega: X = x_i, Y = y_i\}) \equiv p_{ij}, \quad p_{ij} \geq 0, \sum_{i,j} p_{ij} = 1.$$

These probabilities form the *joint distribution* of X and Y. Denote $p_{i\bullet} = \sum_j p_{ij}$ and $p_{\bullet j} = \sum_i p_{ij}$, then random variables X and Y are called *independent* if $p_{ij} = p_{i\bullet} \cdot p_{\bullet j}$, which implies that $E(XY) = E(X)E(Y)$.

The quantity

$$E(X|Y = y_i) := \sum_i x_i \frac{p_{ij}}{p_{\bullet j}}$$

is called the *conditional expectation* of X with respect to the event $\{Y = y_i\}$. The random variable $E(X|Y)$ is called the *conditional expectation* of X with respect to Y if $E(X|Y)$ is equal to $E(X|Y = y_i)$ on every set $\{\omega: Y = y_i\}$. In particular, for indicators $X = I_A$ and $Y = I_B$, we obtain

$$E(X|Y) = P(A|B) = \frac{P(AB)}{P(B)},$$

which is called the *conditional expectation of the event A given the event B*.

We mention some properties of conditional expectations:

1. $E(X) = E\big(E(X|Y)\big)$, in particular, for $X = I_A$ and $Y = I_B$, we have $P(A) = P(B)P(A|B) + P(\Omega \setminus B)P(A|\Omega \setminus B)$;
2. If X and Y are independent, then $E(X|Y) = E(X)$;
3. Since by definition $E(X|Y)$ is a function of Y, then conditional expectation can be interpreted as a *prediction* of X given the information from the "observed" random variable Y.

Finally, for a random variable X with values in $\{0, 1, 2, \dots\}$ we introduce the notion of a *generating function*

$$\phi_X(x) = E(x^X) = \sum_i x^i p_i.$$

We have

$$\phi(1) = 1, \quad \frac{d^k \phi}{dx^k}\Big|_{x=0} = k! p_k$$

and

$$\phi_{X_1 + \cdots + X_k}(x) = \prod_{i=1}^{k} \phi_{X_i}(x)$$

for independent random variables $X_1, \dots, X_k$.

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