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Finite-Difference Methods for Partial Differential Equations

George E. Forsythe and
Wolfgang R. Wasow

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George E. Forsythe
Professor of Mathematics
Stanford University

Wolfgang R. Wasow
Professor of Mathematics
University of Wisconsin

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PREFACE

The astonishingly rapid development of the technology of high-speed computing machines in recent years has been accompanied by a very substantial growth of the mathematical science of numerical analysis. It is no longer possible to do justice to all important aspects of this discipline in one volume. In fact, several branches of the theory, and especially the numerical aspects of differential equations, have become substantial enough to warrant accounts in more than one monograph.

There are many numerical methods for solving partial differential equations. Of these, only one stands out as being universally applicable to both linear and nonlinear problems—the method of finite differences—and we deal exclusively with that method. The literature on difference methods for partial differential equations is growing rapidly. It is widely scattered and differs greatly in viewpoint and character. A definitive presentation of this field will have to wait until the present period of intense development has come to at least a temporary halt. In the meantime, we believe that a connected account of many of the more important results and methods available at this time will serve a useful purpose.

We have tried to keep on a middle ground with respect to the choice of subject matter and the level of presentation. Most of the book ought to be understood by readers with a firm grasp of what is usually taught in a good course in advanced calculus and with some knowledge of matrix theory. Without this prerequisite no real understanding of any but the most elementary aspects of partial differential equations is possible. On the other hand, we do not presuppose a previous knowledge of the theory of partial differential equations, since this would have seriously limited the usefulness of the book. We have excluded topics that have no direct bearing on numerical analysis, such as existence and uniqueness proofs based on finite-difference approximations. At the other extreme, little attempt has been made to serve as a guide for programmers or to include many numerical examples. The numerical solution of partial differential equations is no easy matter. Almost every problem arising out of the physical sciences requires original thought and modifications of existing methods. A general knowledge of the theoretical background and the known methods is almost indispensable for work on such problems, and this is what we have tried to give.

With minor exceptions, the numerical solution of difference equations corresponding to partial differential equations is so enormous a task that it is carried out only with automatic digital computers. For the reader who has had little experience with these machines, Sec. 3 provides some general background about them. In Sec. 25 we discuss in some detail certain algorithms for obtaining and solving difference equations on automatic computers.

The authors were colleagues for some years in the numerical analysis research program at the University of California, Los Angeles, a program founded by the National Bureau of Standards. The book originated in notes prepared by us for a graduate seminar at the university, and was finished at the encouragement of Professors C. B. Tompkins and I. S. Sokolnikoff. The book is directed to several groups of readers: (i) pure and applied mathematical analysts; (ii) programmers of automatic digital computing machines; (iii) engineers, physicists, meteorologists, and others with an interest in using machines to solve partial differential equations; and (iv) graduate students in these fields. Though this was not designed as a textbook, we have used drafts of the book in graduate lectures at our present universities.

Thanks are due the Office of Naval Research, the Office of Ordnance Research of the U. S. Army, the National Science Foundation, and the Mathematics Research Center of the U. S. Army at Madison, Wisconsin, who collectively have supported most of the work on this book at Stanford University, at the University of California, Los Angeles, and at the University of Wisconsin. We wish to thank three graduate students who read much of the manuscript and suggested innumerable improvements: William B. Gragg, Jr., James Ortega, and Betty Jane Stone. Finally, we wish to thank Mrs. Ruthanne Clark, Miss Barbara Spiering, and Mrs. Carolyn Young for their exceptionally responsible typing and other assistance.

George E. Forsythe
Wolfgang R. Wasow

Stanford University,
University of Wisconsin,
December 10, 1959

FINITE–DIFFERENCE METHODS FOR PARTIAL DIFFERENTIAL EQUATIONS

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INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS AND COMPUTERS

SECTION 1. REMARKS ON THE CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATIONS

Let us consider, for the purpose of preliminary orientation, the three differential equations

$$u_{xx} + u_{yy} = 0, \tag{1.1}$$

$$u_{xx} - u_{yy} = 0, \tag{1.2}$$

$$u_{xx} - u_{y} = 0. \tag{1.3}$$

(The subscripts indicate partial differentiation.) They are the prototypes of many important partial differential equations met in the physical applications of mathematics.

As in the theory of ordinary differential equations, one might ask for the "general" solution of such a partial differential equation, but the general solution can be found even more rarely than for ordinary differential equations; and when found, it seldom helps much in answering the questions important to the mathematical physicist. In the applications one is usually concerned with the calculation of a solution which, in addition to the differential equation, satisfies certain subsidiary requirements, such as boundary or initial conditions. For linear (but generally not for nonlinear) ordinary differential equations the desired solution can frequently be found by appropriately determining the arbitrary constants occurring in the general solution. For partial differential equations this is only possible in exceptional cases, one reason being that the general solution now involves arbitrary functions instead of arbitrary constants.

The last remark can be illustrated with equation (1.1). This equation, which is known as *Laplace's equation* and commonly denoted by $\nabla^2 u = 0$ or $\Delta u = 0$, has a close relationship to the theory of analytic functions. Set $z = x + iy$ and let

$$f(z) = u(x, y) + i\, v(x, y)$$

be an analytic function of z. Then u and v are related by the Cauchy-Riemann differential equations

$$u_x - v_y = 0, \qquad u_y + v_x = 0, \tag{1.4}$$

and possess partial derivatives of all orders (Knopp [1945], pp. 28–30). If the first of these equations is differentiated with respect to x and the second with respect to y, it is seen that

$$u(x, y) = \operatorname{Re} f(z) \tag{1.5}$$

is a solution of Laplace's equation. Conversely, we now show that every solution of Laplace's equation is the real part of some analytic function. Let a solution u be given; then equations (1.4) can be solved for v, since the compatibility condition of these two equations is precisely Laplace's equation for u. The quantity $u(x, y) + i\,v(x, y)$ is then (Knopp [1945], p. 30) an analytic function $f(z)$ of the complex variable $z = x + iy$; i.e., (1.5) is valid. Hence (1.5) is the general solution of (1.1).

The solutions of Laplace's equation are frequently called *harmonic* (or potential) functions, and two harmonic functions which are linked by the Cauchy-Riemann equations (1.4) are said to be *conjugate*.

The general solution of equation (1.2) can also be calculated without difficulty, for, if u_{xx} and u_{yy} are continuous, the change of variables

$$\xi = x + y, \qquad \eta = x - y, \qquad u(x, y) = \omega(\xi, \eta)$$

changes (1.2) into

$$\omega_{\xi\eta} = 0,$$

which is solved by

$$\omega = F(\xi) + G(\eta),$$

where F and G are arbitrary differentiable functions; if we require that $\omega_{\xi\eta} = \omega_{\eta\xi}$, there are no other solutions. Hence

$$u(x, y) = F(x + y) + G(x - y) \tag{1.6}$$

is a solution of (1.2), provided F and G are twice differentiable but otherwise arbitrary. [The second derivatives of F and G need not even be continuous for (1.6) to be a solution, as can be verified by inserting (1.6) into (1.2).]

The subsidiary conditions that are imposed on the solution of a differential equation in a problem of mathematical physics vary with the nature of the problem. We give a few extremely simple but typical examples.

(*a*) Consider a rigid wire whose orthogonal projection on the (x, y)-plane is a simple closed curve C. This frame is to contain an ideal elastic membrane of uniform density under uniform tension. Let $u(x, y)$ denote the deflection of this membrane measured from the (x, y)-plane. If u and its derivatives are so small that higher powers of u, u_x, u_y can be neglected by comparison with smaller ones, u can be shown to be a harmonic function in the interior R of C. The values of u on the boundary C are, of course, the prescribed deflection f of the wire frame. Hence u is a solution of the problem

$$\Delta u = 0 \text{ in } R, \qquad u = f \text{ on } C \tag{1.7}$$

(Courant and Hilbert [1953], p. 247). This is frequently called *Dirichlet's problem* for Laplace's equation. For this and other problems, it is important to keep in mind that the solution cannot be expected to have second derivatives, much less to satisfy the differential equation at the boundary points unless f is a rather smooth function. By saying that u *assumes the prescribed boundary values* one means that $u(x, y)$ tends to these values as the point (x, y) approaches the boundary from the interior.

(*b*) A long, straight, narrow rod performs elastic longitudinal vibrations. In a mathematical idealization let the rod be represented by the x-axis, and denote by $u(x, t)$ the deflection from the rest position at time t of the point which, at rest, has the abscissa x. If $u(x, t)$ is small and the units are suitably chosen, u is a solution of the differential equation $u_{tt} - u_{xx} = 0$, the simplest form of the differential equation of wave propagation (Sokolnikoff and Sokolnikoff [1941], p. 367). On physical grounds we expect the values of u at any time to be uniquely determined if the initial deflection $u(x, 0)$ and the initial velocity $u_t(x, 0)$ are prescribed. We are thus led to the problem of finding $u(x, t)$ for $t > 0$, if

$$u_{tt} - u_{xx} = 0 \text{ for } t > 0, \qquad u(x, 0) = f(x), \qquad u_t(x, 0) = g(x), \tag{1.8}$$

where $f(x)$ and $g(x)$ are prescribed arbitrarily, except perhaps for certain smoothness requirements which we do not intend to discuss at this moment. This is an instance of an *initial-value problem*, or "Cauchy's problem," as it is sometimes called.

(*c*) Again we consider a straight, narrow, infinite rod, but this time we let u denote its temperature, whose dependence on x and t we wish to study. We assume that the rod is thermally insulated and that we know the initial distribution of temperature $u(x, 0)$. It is physically plausible that the subsequent temperature distribution $u(x, t)$ is then uniquely determined. One shows easily that the differential equation that ideally governs the flow of heat in the rod is $u_t - u_{xx} = 0$, provided the units are defined properly (Churchill [1941], pp. 15 ff.). The natural initial-value problem in this context is therefore

$$u_t - u_{xx} = 0 \text{ for } t > 0, \qquad u(x, 0) = f(x).$$

In the theory of *ordinary* linear differential equations a given equation can usually be combined in many different ways with subsidiary conditions, which may consist in prescribing data at one, two, or more points. Within certain fairly wide limits such problems will generally have a unique solution, provided the number of conditions matches the order of the differential equation. It is a fact of fundamental importance that this is no longer true for partial differential equations. This had been emphasized and illustrated by Hadamard [1923], pp. 23–44.

He showed, for instance, that Cauchy's problem for Laplace's equation,

$$\Delta u = 0 \text{ for } t > 0, \qquad u(x, 0) = f(x), \qquad u_y(x, 0) = g(x) \tag{1.9}$$

is, in a certain sense, not *well posed.* Let us consider, for example, the special case that $f(x) \equiv 0$, and assume that $u(x, y)$ solves (1.9) in some region bounded below by a segment of the x-axis. Then we can make use of two standard theorems on harmonic functions. The first states that a harmonic function $u(x, y)$ is an analytic function of each of the variables x and y (Sommerfeld [1949], pp. 47–48). The second is a simple consequence of the principle of reflection for analytic functions (Nehari [1952], pp. 183–187). It assures us that a function which is harmonic in a region bounded in part by a straight line segment and zero on this segment can be continued as a harmonic function onto and beyond this segment. Since the derivatives of a harmonic function are themselves harmonic, we conclude from these two facts that $u_y(x, y)$ is a regular analytic function of x for $y = 0$; i.e., $g(x)$ must be analytic. In other words, unless the prescribed function $g(x)$ belongs to the very special class of functions that are analytic, problem (1.9) with $f(x) \equiv 0$ has no solution. It is easy to show, but we shall not do this here, that, if $f(x)$ is not prescribed as identically zero, it also must be analytic if (1.9) is to have a solution.

This severe limitation in the permissible choice of initial values might at first glance be regarded as not very serious. For it is well known that by virtue of Weierstrass' approximation theorem any continuous function can be approximated as closely as we wish by analytic functions, even by polynomials (Courant and Hilbert [1953], pp. 65 ff.). This argument would be valid if close approximation of the boundary values always implied close approximation of the solution for $y > 0$. This, however, is not the case in our present problem. For a counterexample it suffices to consider the initial values

$$f(x) = e^{-\sqrt{n}} \sin nx, \qquad g(x) = 0,$$

where n is a positive integer. It can be easily verified that

$$u(x, y) = e^{-\sqrt{n}} \cosh ny \sin nx$$

is a harmonic function with these initial values. As $n \to \infty$, the initial data tend to zero with all their derivatives, while $u(x, y)$ diverges rapidly for $y \neq 0$.

According to Hadamard, the discontinuous dependence on the initial data precludes by itself any physical meaning for problem (1.9), because physical data are by their nature only approximate. More generally, Hadamard calls a problem of mathematical physics *well posed if its solution exists, is unique, and depends continuously on the data.* It can be shown that the problems (*a*), (*b*), and (*c*) are well posed. If a mathematical problem of physical origin turns out not to be well posed, this usually indicates that the formulation is incorrect or incomplete.

No permutation of the subsidiary conditions in problems (*a*), (*b*), and (*c*) leads to a well-posed problem. We consider one such permutation as a second example of a problem that is not well posed. Let

$$u_{xx} - u_{yy} = 0 \text{ in } R, \qquad u = f \text{ on } C, \tag{1.10}$$

where R is a rectangle whose sides have slopes ± 1. Using again the transformation $\xi = x + y$, $\eta = x - y$, we change the differential equation into $\omega_{\xi\eta} = 0$ and the rectangle R into a rectangle R^* with sides parallel to the ξ, η axes respectively. Suppose that (1.10) possesses a solution. Since ω_ξ does not depend on η, and since ω_η is independent of ξ, the boundary function f must be such that its tangential derivative has equal values at corresponding points on opposite sides of the bounding rectangle. In other words, for arbitrary f, even if severe smoothness restrictions are imposed, problem (1.10) has no solution.

Partial differential equations can be classified according to the type of subsidiary conditions that must be imposed to produce a well-posed problem. In the case of linear differential equations of the second order in two independent variables, this classification is easy to describe. The most general differential equation of this type is

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0, \tag{1.11}$$

with coefficients that are functions of x and y. It is called *elliptic*, *hyperbolic*, or *parabolic* according as the determinant

$$\begin{vmatrix} A & B \\ B & C \end{vmatrix}$$

is positive, negative, or zero. This classification depends in general on the region of the (x, y)-plane under consideration. The differential equation $xu_{xx} + u_{yy} = 0$, for instance, is elliptic for $x > 0$, hyperbolic for $x < 0$, and parabolic for $x = 0$.

Each of the three simple equations we have been discussing in this section is an example of one of these types: Laplace's equation is elliptic, the equation of wave propagation is hyperbolic, and the equation of heat flow is parabolic. It can be shown that the subsidiary conditions imposed by us in each case will generate well-posed problems also when combined with more general differential equations of the respective type but not when combined with differential equations of any other type.

For differential equations in more than two variables, for systems, and for nonlinear differential equations, useful definitions of the concepts of elliptic, hyperbolic, and parabolic character can also be given. We shall introduce these as the need arises.

Recently, Hadamard's position that only well-posed problems are physically relevant has been questioned by several mathematicians. In the case of certain initial-value problems for elliptic differential equations of physical origin, numerical schemes have been suggested which should approximate the exact solution even though the latter depends in discontinuous fashion on the initial values. But these investigations are as yet too incomplete to warrant a description herein.

SECTION 2. SYSTEMS AND SINGLE EQUATIONS

Every single differential equation of order higher than one can be written as a system of first-order equations. This is rather obvious. One way of doing it is to introduce all derivatives of the dependent variable, except those of highest order, as new unknown functions. Thus $u_{xx} - u_{yy} = 0$ is equivalent to the system

$$u_x = p, \qquad u_y = q, \qquad p_x - q_y = 0$$

for the three functions u, p, q.

For ordinary differential equations a converse of this statement is also true: From a system of n first-order equations for n functions satisfying certain mild regularity conditions one can derive one differential equation of order n containing only one of these unknown functions. For instance, from the two simultaneous equations $f(x, u, v, u', v') = 0, g(x, u, v, u', v') = 0$, the unknown function v and its derivatives can be eliminated by first solving for v and v', which leads to two equations of the form

$$v = \phi(x, u, u'), \qquad v' = \psi(x, u, u').$$