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Global Aspects of Complex Geometry

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With 15 Figures

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Global Aspects of Complex Geometry

Preface

Over the period 2000–2006 the Deutsche Forschungsgemeinschaft sponsored a special Schwerpunkt programme, entitled "Global Methods in Complex Geometry".

The articles of this volume grew out of this programme and document some of the scientific activity performed in the realm of the Schwerpunkt. They also aim at giving a broader overview of recent developments in various directions of Complex Geometry such as

- Low-dimensional geometry: surfaces of general type, Fano threefolds, Calabi-Yau threefolds;
- moduli spaces and families of varieties over curves;
- Hodge theory, motivic cohomology and characteristic p-geometry;
- moment maps and group actions on flag manifolds;
- geometry of singular varieties: vector fields, equisingular families and vector bundles;
- geometry of rational curves and pseudo-effective line bundles.

The articles are devoted to a broad spectrum of topics, which range from purely algebraic to complex-analytic aspects of our subject.

The participants of the Schwerpunkt would like to thank the Deutsche Forschungsgemeinschaft for its generous support.

Bayreuth, Essen, Bochum, Hannover, June 2006

Fabrizio Catanese, Hélène Esnault, Alan Huckleberry, Klaus Hulek, Thomas Peternell

Contents

Complex Surfaces of General Type: Some Recent Progress

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Introduction

In this article we shall give an overview of some recent developments in the theory of complex algebraic surfaces of general type.

After the *rough* or *Enriques-Kodaira* classification of complex (algebraic) surfaces, dividing compact complex surfaces in four classes according to their Kodaira dimension $-\infty$, 0, 1, 2, the first three classes nowadays are quite well understood, whereas even after decades of very active research on the third class, the class of surfaces *of general type*, there is still a huge number of very hard questions left open. Of course, we made some selection, which is based on the research interest of the authors and we claim in no way completeness of our treatment. We apologize in advance for omitting various very interesting and active areas in the theory of surfaces of general type as well as for not being able to mention all the results and developments which are important in the topics we have chosen.

Complex surfaces of general type come up with certain (topological, birational) invariants, topological as for example the topological Euler number e and the self intersection number of the canonical divisor K^2 of a minimal surface, which are linked by several (in-) equalities. In the first chapter we will summarize the classically known inequalities, which force surfaces of general type in a certain region of the plane having K^2 and e as coordinates, and we shall briefly comment on the so-called *geography* problem, whether,

⋆ The present work was performed in the realm of the SCHWERPUNKT "Globale Methoden in der komplexen Geometrie", and was also supported by a VIGONI-DAAD Program. A first draft of this article took origin from the lectures by the second author at the G.A.C. Luminy Meeting, october 2005: thanks to the organizers!

given numerical invariants lying in the admissible range, i.e., fulfilling the required inequalities, does there exist a surfaces having these invariants. We shall however more broadly consider the three classical invariants K^2, p_g, q, which determine the other invariants $\chi := 1 - q + p_g, e = 12\chi - K^2$.

An important new inequality, which Severi tried without success to establish, and which has been attacked for many years with partial results by several authors, asserts that a surface of maximal Albanese dimension satisfies the inequality $K^2 \geq 4\chi$. We will report on Pardini's surprisingly simple proof of this so-called Severi's conjecture (cf. [Par05]).

The study of the pluricanonical maps is an essential technique in the classification of surface of general type. The main results concerning the m-canonical maps with $m \geq 3$ go back to an earlier period and we refer to [Cat87b] for a report on them.

We will report in the second chapter on recent developments concerning the bicanonical map; we would like to mention Ciliberto's survey (cf. [Cil97]) on this topic for the state of art ten years ago. Here instead, we combine a discussion of this topic with the closely intertwined problem of classification of surfaces with low values of the numerical invariants.

In the third chapter we report on surfaces of general type with geometric genus p_g equal to four, a class of surfaces whose investigation was started by Federigo Enriques (cf. chapter VIII of his book 'Le superficie algebriche', [Enr49]).

By Gieseker's theorem we know that for fixed K^2 and χ there exists a quasi projective coarse moduli space $\mathcal{M}_{K^2,\chi}$ for the birational equivalence classes of surfaces of general type. It is a very challenging problem to understand the geometry of these moduli spaces even for low values of the invariants. The case $p_g = 4$ is studied via the behaviour of the canonical map. While it is still possible to divide the moduli space into various locally closed strata according to the behaviour of the canonical map, it is very hard to decide how these strata patch together.

Using certain presentations of Gorenstein rings of codimension 4 introduced by M. Reid and D. Dicks, which arrange the defining equations as Pfaffians of certain matrices with many symmetries in such a way that these equations behave well under deformation, it is possible to exhibit explicit deformations, which allow to "connect" certain irreducible components of the moduli space.

Inspired by a construction of A. Beauville of a surface with $K^2 = 8$, $p_g = q = 0$, the second author defined Beauville surfaces as surfaces which are rigid and which admit an unramified covering which is isomorphic to a product of curves of genus at least 2. In this case the moduli space of surfaces orientedly homeomorphic to a given surface consists either of a unique real point, or of a pair of complex conjugate points corresponding to complex conjugate surfaces.

These surfaces, and the more general surfaces isogenous to a product, not only provide cheap counterexamples to the Friedman-Morgan speculation (which will be treated more extensively in the sixth section of this article), but provide also a wide class of surfaces quite manageable in order to test conjectures, and offer also counterexamples to various problems. The ease with which one can handle these surfaces is based on the fact that these surfaces are determined by "discrete" combinatorial data.

Beauville surfaces, their relations to group theory and to Grothendieck's theory of 'Dessins d'enfants' will be discussed in the fourth chapter.

It is a very difficult and very intriguing problem to decide whether two algebraic surfaces, which are not deformation equivalent, are in fact diffeomorphic.

The theory of Lefschetz fibrations provides an algebraic tool to prove that two surfaces are diffeomorphic. By a theorem of Kas (which holds also in the symplectic context) two Lefschetz fibrations are diffeomorphic if and only if their corresponding factorizations of the identity in the mapping class group are equivalent under the equivalence relation generated by Hurwitz moves and by simultaneous conjugation. We outline the theory, which was used with success in [CW04] in chapter five, which we end with a brief report on the status of two very old conjectures by Chisini concerning cuspidal curves and algebraic braids.

As already mentioned before, one of the fundamental problems in the theory of surfaces of general type is to understand their moduli spaces, in particular the connected components which parametrize the deformation equivalence classes of minimal surfaces of general type. By a classical result of Ehresmann, two deformation equivalent algebraic varieties are diffeomorphic. The other direction, i.e., whether two diffeomorphic minimal surfaces of general type are indeed in the same connected component of the moduli space, was an open problem since the eighties. We discuss in the last chapter the various counterexamples to the Friedman-Morgan speculation, who expected a positive answer to the question (unlike the second author, cf. [Kat83]).

Moreover, we briefly report on another equivalence relation introduced by the second author, the so-called quasi étale-deformation (Q.E.D.) equivalence relation, i.e., the equivalence relation generated by birational equivalence, by quasi étale morphisms and by deformation equivalence. For curves and surfaces of special type two varieties are Q.E.D. equivalent if and only if they have the same Kodaira dimension, whereas there are infinitely many surfaces of general type, which are pairwise not Q.E.D. equivalent.

1 Old and New Inequalities

1.1 Invariants of Surfaces

Let X be a compact complex manifold and let Ω^n_X be its canonical bundle, i.e., the line bundle of holomorphic $n-$forms (usually denoted by ω_X, since it is a dualizing sheaf in the sense of Serre duality). A corresponding canonical divisor is usually denoted by K_X.

To X one associates its *canonical ring*

$$R(X) := \oplus_{m\geq 0} H^0(\omega_X^{\otimes m}).$$

The trascendency degree over $\mathbb{C}$ of this ring leads to

- the *Kodaira dimension* $\kappa(X) := tr(R(X)) - 1$,

if $R(X) \neq \mathbb{C}$, otherwise $\kappa(X) := -\infty$. The Kodaira dimension is invariant under deformation (by Siu's theorem [Siu02], generalizing Iitaka's theorem for surfaces) and can assume the values $-\infty, 0, \ldots, n = \dim X$.

Definition 1. *X is said to be* of general type *if the Kodaira dimension is maximal, $\kappa(X) = \dim X$.*

We are interested in the case of *surfaces*, i.e., of manifolds of dimension 2, of general type.

The three principal invariants under deformations for the study of these surfaces are

- the self intersection of the canonical class K_S^2 of a minimal model,
- the geometric genus $p_g := h^0(\omega_X)$ and
- the irregularity $q := h^1(\mathcal{O}_S) = h^0(\Omega^1_S)$.

The equality $h^1(\mathcal{O}_S) = h^0(\Omega^1_S)$ follows by Hodge theory since every algebraic surface is projective.

The invariants we have introduced, with the exception of K_S^2, are not only deformation invariants but also birational invariants.

Definition 2. *A smooth surface S is called* minimal *(or* a minimal model*) iff it does not contain any exceptional curve E of the first kind (i.e. $E \cong \mathbb{P}^1$, $E^2 = -1$).*

Every surface can be obtained by a minimal one (its "minimal model") after a finite sequence of blowing ups of smooth points; this model is moreover unique if $\kappa(S) \geq 0$ (see III.4.4, III.4.5 and III.4.6 of [BHPV04]). Thus, every birational class of surfaces of general type contains exactly one minimal surface, and one classifies surfaces of general type by studying their minimal models. To each minimal surface of general type we will associate its numerical

- type (K_S^2, p_g, q),

a triple of integers given by the three invariants introduced above.

In fact these determine all other classical invariants, as

- the Euler-Poincaré characteristic of the trivial sheaf $\chi(\mathcal{O}_S) = 1 - q + p_g$;
- the topological Euler characteristic $e(S) = c_2(S) = 12\chi(\mathcal{O}_S) - K_S^2$;
- the plurigenera $P_m(S) := h^0(\omega_X^{\otimes m}) = \chi(\mathcal{O}_S) + \binom{m}{2} K_S^2$.

The epression for c_2 is a classical theorem of M. Noether, and the expression for the plurigenera follows by Riemann-Roch and by Mumford's vanishing theorem.

By the theorems on pluricanonical maps (cf. [Bom73]), minimal surfaces S of general type with fixed invariants are birationally mapped to normal surfaces X in a fixed projective space of dimension $P_5(S) - 1$. X is uniquely determined, is called the *canonical model* of S, and is obtained contracting to points all the (-2)-curves of S (curves $E \cong \mathbb{P}^1$, with $E^2 = -2$).

Let us recall Gieseker's theorem

Theorem 1 (Gieseker [Gie77]). *There exists a quasi-projective coarse moduli scheme for canonical models of surfaces of general type S with fixed K_S^2 and $c_2(S)$.*

In particular, we can consider the subscheme $\mathcal{M}_{K_S^2, p_g, q}$ corresponding to minimal surfaces of general type of type (K_S^2, p_g, q). By the above theorem, it is a quasi projective scheme, in particular, it has finitely many irreducible components.

It is a dream ever since to completely describe $\mathcal{M}_{K_S^2, p_g, q}$ for as many types as possible.

1.2 Classical Inequalities and Geography

Obviously the first question is: for which values of (K_S^2, p_g, q) is $\mathcal{M}_{K_S^2, p_g, q}$ non empty?

For example, it is clear that $p_g(S)$ and $q(S)$ are always nonnegative, since they are dimensions of vector spaces.

In fact much more is known. In the following table we collect the well known classical inequalities holding among the invariants of minimal surfaces of general type:

	$K_S^2 \geq 1$	$\chi \geq 1$
(N)	$K_S^2 \geq 2p_g - 4$	or the weaker $K_S^2 \geq 2\chi(\mathcal{O}_S) - 6$
(D)	if $q > 0$, $K_S^2 \geq 2p_g$	or the weaker if $q > 0$, $K_S^2 \geq 2\chi(\mathcal{O}_S)$
(MY)	$K_S^2 \leq 9\chi$	

We have labeled by (N)= Noether, (D) = Debarre, (MY) = Miyaoka-Yau the rows, corresponding to the names of the inequalities ([Deb82], [Deb83], [Miy77], [Yau78], see also [BHPV04], chap. 7).

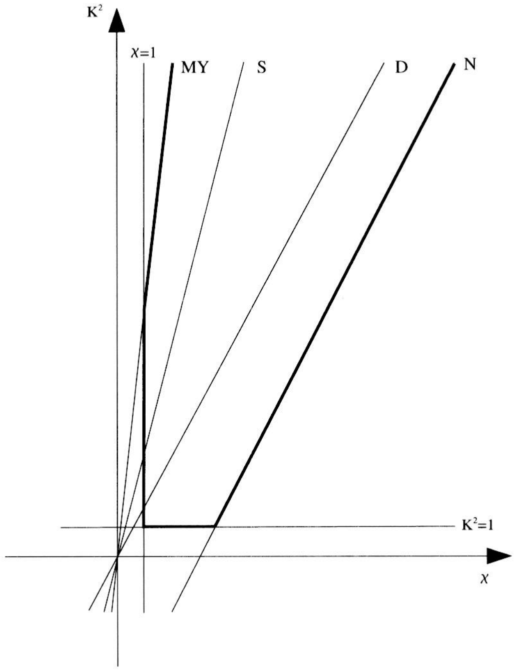

Fig. 1. The geography of minimal surfaces of general type

In figure 1 we have drawn the limit lines (i.e., where equality holds) of the various inequalities in the (χ, K_S^2)-plane.

The above listed inequalities show that the pair of invariants χ, K_S^2 of a minimal surface of general type gives a point with integral coordinates in the convex region limited by the "bold" piecewise linear curve. Moreover, if $q > 0$ this point cannot be at the "right" of the line D.

We drew one more line in our picture, labeled by S. This is the Severi line $K^2 = 4\chi$, i.e., the equality case of the Severi inequality $K^2 \geq 4\chi \Leftrightarrow K^2 \geq \frac{1}{2}e$, which will be discussed in detail at the end of this section.

1.3 Surfaces Fibred over a Curve

An important method for the study of surfaces of general type is to consider relatively minimal fibrations of surfaces over curves $f : S \to B$.

Definition 3. *A fibration $f : S \to B$ is a surjective morphism with connected fibres. We are interested in the case of fibrations* of surfaces to curves, *meaning that in this paper S and B will always be smooth compact complex manifolds of respective dimensions 2 and 1.*

The fibration is said to be relatively minimal *if f does not contract any rational curve of self intersection -1 to a point.*

One denotes

- by b the genus of the base curve B;
- by g the genus of a general fibre.

To avoid confusion, let us point out that a fibration is called rational or irrational according to the genus b of the base being 0 or > 0. On the other hand, the genus of the fibration is the genus g of the fibre. For example, if we say f is a genus 2 rational fibration, we intend that $g = 2$ and $b = 0$.

The classical way of saying: a genus b pencil of curves of genus g is however still the most convenient way to describe a fibration.

To a relatively minimal fibration f one associates

- its relative canonical bundle $\omega_{S|B} := \omega_S \otimes f^*(\omega_B^\vee)$ and
- the sheaves $(\forall n \geq 0)$ $V_n := f_*(\omega_{S|B}^{\otimes n})$.

The sheaves V_n are vector bundles (i.e., locally free sheaves) with very nice properties.

Theorem 2 (Fujita [Fuj78a], [Fuj78b]). *The vector bundles V_n are* semi-positive, *i.e., every locally free quotient of it has nonnegative degree.*

To be more precise, V_1 is a direct sum of an ample vector bundle with $q(S) - b$ copies of the trivial bundle and with some undecomposable stable degree 0 vector bundle without global sections. Zucconi [Zuc97] proved moreover that if one of those stable bundles has rank 1, then it is a torsion line bundle.

For $n \geq 2$ we have:

Theorem 3 (Esnault-Viehweg [EV90]). *$\forall n \geq 2$ the vector bundle V_n is ample unless f has constant moduli, which means that all the smooth fibres are isomorphic.*

Since $R^1 f_* \omega_{S|B} = \mathcal{O}_B$ by relative duality, and $R^1 f_* \omega_{S|B}^{\otimes n} = 0 \ \forall n \geq 2$ by the assumption of relative minimality, one can compute the Euler characteristic of V_n by Riemann-Roch, and consequently its degree.

We introduce the following invariants of the fibration f:

- the self intersection of the relative canonical divisor

$$K_f^2 := \omega_{S|B} \cdot \omega_{S|B} = K_S^2 - 8(g-1)(b-1),$$

- the Euler characteristic of the relative canonical divisor

$$\chi_f = \chi(\omega_{S|B}) = \chi(\mathcal{O}_S) - (g-1)(b-1),$$

- its slope $\lambda(f) := K_f^2 / \chi_f$.

The slope is clearly defined only for $\chi_f \neq 0$, or equivalently (as we will see soon) if the fibration is not a holomorphic bundle.

The above mentioned computation gives

$$\deg V_n = \chi_f + \frac{n(n-1)}{2} K_f^2$$

and since by Fujita's theorem these numbers are nonnegative this gives the two inequalities $K_f^2 \geq 0$ and $\chi_f \geq 0$ respectively known as Arakelov's inequality (cf. [Ara71]) and Beauville's inequality (cf. [Bea82]).

In fact, we have the following list of inequalities

$$\begin{array}{ll} (A) & K_f^2 \geq 0 \ , \ i.e., \ K_S^2 \geq 8(g-1)(b-1), \\ (B) & \chi_f \geq 0 \ , \ i.e., \ \chi(\mathcal{O}_S) \geq (g-1)(b-1), \\ (ZS) & c_2(S) \geq 4(b-1)(g-1), \\ (NN) & q \leq b+g, \\ (X) & 4 - \frac{4}{g} \leq \lambda(f) \leq 12. \end{array}$$

Here the meaning of the labeling is the following: (A) = Arakelov's inequality, (B) = Beauville' inequality, (X) = Xiao's inequality (also known as slope inequality), (NN) = no name's inequality, (ZS) = Zeuthen-Segre. A proof of those inequalities can be found in [Bea82] with the exception of the slope inequality, proved in [Xia87] (see also [CH88] in the semistable case).

The equality cases of the first 4 inequalities are well described:

- if equality holds in (A), f has constant moduli;
- equality holds in (B) $\Leftrightarrow$ f has constant moduli and is smooth;
- for $g \geq 2$, equality holds in (ZS) $\Leftrightarrow$ f is smooth;
- $q = b + g \Leftrightarrow f$ is birationally equivalent to the projection of a product $B \times F$ to the first factor.

In particular, we see that the slope is defined whenever the fibration is not a holomorphic bundle, since the denominator χ_f vanishes iff equality holds in Beauville's inequality.

An important consequence is the following

Theorem 4 (Beauville). *If X is a minimal surface of general type, then $p_g \geq 2q - 4$. Moreover, if $p_g = 2q - 4$, then S is a product of a curve of genus 2 with a curve of genus $q - 2$.*

Note for later use (see next section) the following

Corollary 1. *If $p_g = q$ (i.e., if $\chi(\mathcal{O}_S) = 1$), then $p_g = q \leq 4$. Moreover, minimal surfaces of general type with $p_g = q = 4$ are exactly the products of two genus 2 curves.*

Proof of theorem 4. The standard wedge product on 1−forms induces a natural map

$$\wedge : \Lambda^2 H^0(\Omega_S^1) \to H^0(\Omega_S^2)$$

Recall that $q = \dim H^0(\Omega^1_S)$, $p_g = \dim H^0(\Omega^2_S)$. Let us assume $p_g \leq 2q - 4$.

By a dimension count, if $p_g \leq 2q - 4$, the projective linear subspace of $\mathbb{P}(\Lambda^2 H^0(\Omega^1_S))$ corresponding to the kernel of the above map must intersect the Plücker embedding of the Grasmannian $G_2(H^0(\Omega^1_S))$ (which has dimension $2q - 4$), and therefore there are two linearly independent 1−forms ω_1 and ω_2 such that the following holomorphic two form is identically zero: $\omega_1 \wedge \omega_2 \equiv 0$.

By the theorem of Castelnuovo-De Franchis there is a fibration $f : S \to B$ with base of genus $b \geq 2$, and two holomorphic 1−forms $\alpha_1, \ \alpha_2 \in H^0(\Omega^1_B)$ such that $f^*\alpha_i = \omega_i$. Since S is of general type, also $g \geq 2$.

Then

$$\chi_f \geq 0 \Rightarrow \chi(\mathcal{O}_S) \geq (b-1)(g-1) = (b-2)(g-2) + b + g - 3 \geq q - 3.$$

So we have $1 - q + p_g \geq q - 3 \Leftrightarrow p_g \geq 2q - 4$.

If $p_g = 2q - 4$, all inequalities are equalities and then, since $q = b + g$ and $(b-2)(g-2) = 0$, S is a product of two curves of genus at least 2, and one of the two must have genus exactly 2.

□

1.4 Severi's Inequality

We recall that the Albanese variety $Alb(X)$ of a compact Kähler manifold X is the cokernel of the natural map

$$\int : H_1(X, \mathbb{Z}) \to H^0(\Omega^1(X))^\vee$$

defined by integrating 1−forms on 1-cycles.

The Albanese morphism

$$\alpha : X \to Alb(X)$$

is defined (up to translations in $Alb(X)$) by fixing a point $p_0 \in X$, and by associating to each point $p \in X$ the class in $Alb(X)$ of $\int_{p_0}^{p}$, where the integral is taken along any path between p_0 and p.

Recall that, if X is projective (as any surface of general type), $Alb(X)$ is an abelian variety (of dimension q).

The Albanese morphism is a powerful tool for studying *irregular* surfaces ($q > 0$) and in particular:

Definition 4. *A variety X is called of* maximal Albanese dimension *if the image of the Albanese morphism has the same dimension as X.*

This is the general case for surfaces, since otherwise the Albanese morphism is a fibration onto a smooth curve of genus q. We see then that for surfaces maximal Albanese dimension is equivalent to the non existence of a genus q pencil.

We can now state the theorem known as *Severi's inequality*

Theorem 5 (Pardini [Par05]). *If S is a smooth complex minimal surface of maximal Albanese dimension, then $K_S^2 \geq 4\chi$.*

This theorem was proved only very recently by R. Pardini, but it has a long story, which we briefly sketch in the following.

Severi's Conjecture

The inequality takes its name from F. Severi, since he was the first to claim the result in the 30's [Sev32].

His proof turned out to be wrong, as was pointed out in [Cat83], since it was based on the assertion that a surface with irregularity q either contains an irrational genus q fibration, or the sections of $H^0(\Omega_S^1)$ have no common zero. Counterexamples were given in [Cat84], where there were constructed bidouble covers $S \to X$ of any algebraic surface with, among other properties, $q(S) = q(X)$. If X has no irrational pencils, since the Albanese map of S factors through the cover, then also S has no irrational pencils. But any ramification point of the cover is a base point for $H^0(\Omega_S^1)$.

Therefore Severi's inequality was posed in [Cat83] as *Severi's conjecture*, a conjecture on surfaces of general type, since for surfaces with $\kappa(S) \leq 1$ it is a straightforward consequence of the Enriques-Kodaira classification. It had also been posed as a conjecture by M. Reid (conj. 4 in [Rei79]) who proved the weaker $K_S^2 \geq 3\chi$.

Proofs in Special Cases

In the 80's, Xiao's work on surfaces fibred over a curve was mainly motivated by Severi's conjecture. In [Xia87] he proved the slope inequality and Severi's conjecture for surfaces having an irrational pencil.

In the 90's Konno [Kon96] proved the conjecture in the special case of *even* surfaces, i.e., surfaces whose canonical class is 2− divisible in the Picard group.

Finally, at the end of the 90's, Manetti [Man03] could prove the inequality for surfaces of general type whose canonical bundle is ample.

Manetti's Proof

Manetti considers the tautological line bundle L of the $\mathbb{P}^1$−bundle $\pi : \mathbb{P}(\Omega_S^1) \to S$; standard computations give

$$3(K_S^2 - 4\chi) = L^2 \cdot (L + \pi^* K_S).$$

Then, using the fact that Ω_S^1 is generically globally generated, he can write the right hand side of the above equation as $2K_S E + (L + \pi^* K_S)C$ for an effective 1−cycle C in $\mathbb{P}(\Omega_S^1)$, and where E is the maximal effective divisor